

A STRUCTURE THEOREM FOR SEMI-PARABOLIC HÉNON MAPS

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ABSTRACT. Consider the parameter space $\mathcal{P}_\lambda \subset \mathbb{C}^2$ of complex Hénon maps

$$H_{c,a}(x, y) = (x^2 + c + ay, ax), \quad a \neq 0$$

which have a semi-parabolic fixed point with one eigenvalue $\lambda = e^{2\pi i p/q}$. We give a characterization of those Hénon maps from the curve \mathcal{P}_λ that are small perturbations of a quadratic polynomial p with a parabolic fixed point of multiplier λ . We prove that there is an open disk of parameters in \mathcal{P}_λ for which the semi-parabolic Hénon map has connected Julia set J and is structurally stable on J and J^+ . The Julia set J^+ has a nice local description: inside a bidisk $\mathbb{D}_r \times \mathbb{D}_r$ it is a trivial fiber bundle over J_p , the Julia set of the polynomial p , with fibers biholomorphic to \mathbb{D}_r . The Julia set J is homeomorphic to a quotiented solenoid.

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1. INTRODUCTION

A Hénon map is a polynomial automorphism of \mathbb{C}^2 and can be written as

$$H_{c,a}(x, y) = (x^2 + c + ay, ax), \quad \text{for } a \neq 0,$$

where a and c are complex parameters. In this parametrization, the Hénon map has constant Jacobian $-a^2$. In order to study the dynamics of polynomial automorphisms of \mathbb{C}^2 we need to understand their behavior under forward and backward iterations. The dynamical objects that we need to analyze are the sets K^\pm (the set of points with bounded forward/backward orbits) and their topological boundaries $J^\pm = \partial K^\pm$. The set $J = J^+ \cap J^-$ is the analogue of the one-dimensional Julia set for polynomials.

We say that the Hénon map is hyperbolic if it is hyperbolic on its Julia set J . If $H_{c,a}$ is hyperbolic and $|a| < 1$ then the interior of K^+ consists of the basins of attraction of finitely many attractive periodic points [BS1]. Each basin of attraction is a Fatou-Bieberbach domain (a proper subset of \mathbb{C}^2 , biholomorphic to \mathbb{C}^2). The common boundary of the basins is the set J^+ [BS1]. The set J^+ is where the most interesting chaotic behavior takes place. For hyperbolic Hénon maps, periodic points are dense in J and the map is structurally stable on J [BS1]. When $x \mapsto x^2 + c$ is a hyperbolic polynomial, the Hénon map $H_{c,a}$ is also hyperbolic for small values of a . Hyperbolic Hénon maps that come from perturbations of hyperbolic polynomials are very well understood, by work of Hubbard and Oberste-Vorth [HOV1], [HOV2] and Fornæss and Sibony [FS]. However, there is very little known about Hénon maps which are not hyperbolic.

In this paper, we study Hénon maps with a semi-parabolic fixed point (or cycle). A fixed point of $H_{c,a}$ is called semi-parabolic if the derivative of $H_{c,a}$ at the fixed point has two eigenvalues $\lambda = e^{2\pi ip/q}$ and μ , with $|\mu| < 1$. For clarity and simplicity of exposition, we will call a Hénon map semi-parabolic if it has a semi-parabolic fixed point.

Unlike hyperbolic Hénon maps, which exhibit structural stability, semi-parabolic Hénon maps are not expected to be structurally stable. The general assumption is that bifurcations will occur as we perturb from a semi-parabolic Hénon map. Bedford, Smillie, and Ueda show in [BSU] some of the complications that can arise by describing the phenomenon of “semi-parabolic implosion” in \mathbb{C}^2 (discontinuity of J and J^+ on the parameters). We prove that there are classes of semi-parabolic Hénon maps that are structurally stable on the sets J and J^+ inside a parametric region of codimension one in \mathbb{C}^2 . We give a complete characterization of the dynamics of these Hénon maps. In particular, we show that J is homeomorphic to a solenoid with identifications, hence it is connected.

This parametric region of structural stability will be obtained by considering appropriate perturbations in \mathbb{C}^2 of a polynomial with a parabolic fixed point of multiplier $\lambda = e^{2\pi ip/q}$. The set of parameters $(c, a) \in \mathbb{C}^2$ for which the Hénon map $H_{c,a}$ has a fixed point with one eigenvalue λ is an algebraic curve \mathcal{P}_λ in \mathbb{C}^2 . The parameter $c = c(a)$ is a function of a and the parametric line $a = 0$ intersects the curve \mathcal{P}_λ at a point c_0 . The polynomial $p(x) = x^2 + c_0$ has a parabolic fixed point of multiplier λ . Let J_p be the Julia set of the parabolic polynomial p .

In this article, we will describe the class of semi-parabolic Hénon maps $H_{c,a}$ where (c, a) lies in a small disk around 0 inside the curve \mathcal{P}_λ . Consider $r > 3$ and let V denote the bidisk $\mathbb{D}_r \times \mathbb{D}_r$ throughout this section. We can now state the following theorem.

Theorem 1.1 (Structure Theorem). *Let $p(x) = x^2 + c_0$ be a polynomial with a parabolic fixed point of multiplier $\lambda = e^{2\pi i p/q}$. There exists $\delta > 0$ such that for all parameters $(c, a) \in \mathcal{P}_\lambda$ with $0 < |a| < \delta$ there exists a homeomorphism*

$$\Phi : J_p \times \mathbb{D}_r \rightarrow J^+ \cap V$$

which makes the diagram

$$\begin{array}{ccc} J_p \times \mathbb{D}_r & \xrightarrow{\Phi} & J^+ \cap V \\ \psi \downarrow & & \downarrow H_{c,a} \\ J_p \times \mathbb{D}_r & \xrightarrow{\Phi} & J^+ \cap V \end{array}$$

commute, where

$$\psi(\zeta, z) = \left(p(\zeta), a\zeta - \frac{a^2 z}{p'(\zeta)} \right).$$

The map ψ depends on a , but we will show in Lemmas 12.7 and 12.8 that all maps ψ are conjugate to each other, for sufficiently small $0 < |a| < \delta$. Thus it does not matter which one we use and we can assume that the model map is $\psi(\zeta, z) = \left(p(\zeta), \epsilon\zeta - \frac{\epsilon^2 z}{p'(\zeta)} \right)$, for some $\epsilon > 0$ independent of a . The function ψ is a solenoidal map in the sense of [HOV1]; it behaves like angle-doubling in the first coordinate, and contracts strongly in the second coordinate.

Theorem 1.1 shows that $J^+ \cap V$ is a trivial fiber bundle over J_p , the Julia set of the parabolic polynomial $p(x) = x^2 + c_0$, with fibers biholomorphic to \mathbb{D}_r . The set J^+ is laminated by Riemann surfaces isomorphic to \mathbb{C} . In fact, the current μ^+ supported on J^+ defined by Bedford and Smillie in [BS1] is laminar.

Theorem 1.2 (Model for J). *The Julia set J for the Hénon map is homeomorphic to a quotiented solenoid*

$$J \simeq \bigcap_{n \geq 0} \psi^{\circ n}(J_p \times \mathbb{D}_r),$$

hence connected. Moreover $J = J^*$, where J^* is the closure of the saddle periodic points.

We describe J as the quotient of a topological solenoid; it is easy to pass from the topological model to a combinatorial description and see that J is equivalent to a dyadic solenoid as in [HOV2] (as a projective limit of J_p under the polynomial p).

In Corollary 12.9.1 we establish that $J = J^*$, the closure of the saddle periodic points. This was not known for this particular class of Hénon maps and it is still an open question whether $J = J^*$ in general. It was shown to be true if J is hyperbolic [BS1].

Let $f : X \rightarrow X$ be an open, injective map from a space X to itself. Define the inductive limit as $\varinjlim(X, f) := X \times \mathbb{N} / \sim$, where the equivalence relation is defined by $(x, n) \sim (f(x), n+1)$. In this setting, the inductive limit is an increasing union of sets homeomorphic to X , so locally it looks like X . The limit space comes with a natural bijective map $\tilde{f} : \varinjlim(X, f) \rightarrow \varinjlim(X, f)$ given by $(x, n) \mapsto (f(x), n)$.

Passing to the inductive limit as done by Hubbard and Oberste-Vorth in [HOV2] we get a global model for J^+ .

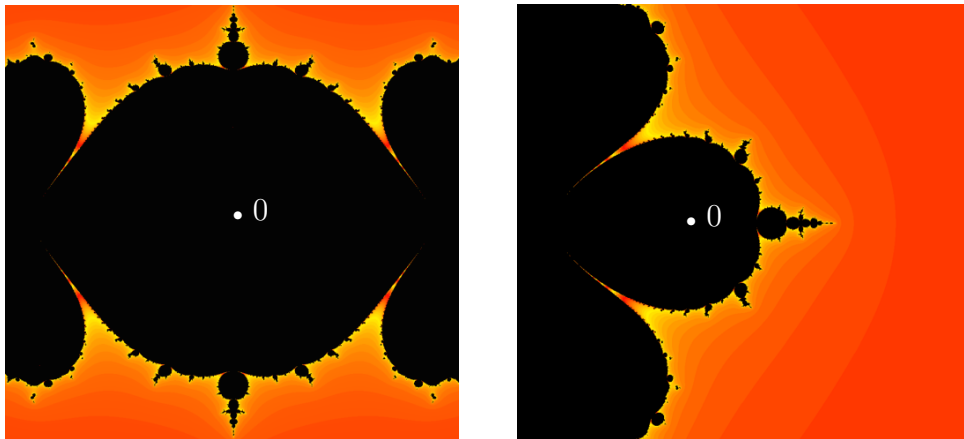


Figure 1. A parameter plot inside the curve \mathcal{P}_{-1} . In both pictures the large region in the center contains the disk $|a| < \delta$. The black region represents (a rough approximation of) the set of parameters $(c, a) \in \mathcal{P}_{-1}$ for which J is connected. The picture on the left is a double cover of the picture on the right. Both pictures were generated using FractalStream. LEFT: The Hénon map is written as $H_{c,a}(x, y) = (x^2 + c + ay, ax)$. RIGHT: The Hénon map is written in the standard form $H_{c,a}(x, y) = (x^2 + c - ay, x)$.

Theorem 1.3 (Model for J^+). *The map Φ extends naturally to a homeomorphism $\check{\Phi}$ and the following diagram*

$$\begin{array}{ccc}
 \varinjlim (J_p \times \mathbb{D}_r, \psi) & \xrightarrow{\check{\Phi}} & J^+ \\
 \check{\psi} \downarrow & & \downarrow H_{c,a} \\
 \varinjlim (J_p \times \mathbb{D}_r, \psi) & \xrightarrow{\check{\Phi}} & J^+
 \end{array}$$

commutes.

As a consequence of the previous theorems, we get that the family of semi-parabolic Hénon maps $P_\lambda \ni (c, a) \rightarrow H_{c,a}$ is a structurally stable family on J and J^+ for $|a| < \delta$. By structural stability on J and J^+ we understand the following:

Theorem 1.4 (Stability). *If (c_1, a_1) and (c_2, a_2) belong to \mathcal{P}_λ and if $0 < |a_i| < \delta$ then J_{c_1, a_1} is homeomorphic to J_{c_2, a_2} and $(H_{c_1, a_1}, J_{c_1, a_1})$ is conjugate to $(H_{c_2, a_2}, J_{c_2, a_2})$. The same is true for J^+ instead of J .*

Let $\lambda = 1$ and consider perturbations of the parabolic polynomial $p(x) = x^2 + 1/4$ (the root of the main cardioid of the Mandelbrot set) inside \mathcal{P}_1 . The Julia set J_p of this polynomial is a Jordan curve and therefore Theorem 1.2 implies that the Julia set $J_{c,a}$ of the Hénon map is homeomorphic to a solenoid. Together with Theorem 1.4 this gives a positive answer to some questions of Bedford (Questions 1 and 2 in [B]).

Moreover, the set $J_{c,a}^+$ is homeomorphic to a 3-sphere with a dyadic solenoid removed, for all $(c, a) \in \mathcal{P}_1$ and a sufficiently small [R].

In order to have nontrivial identifications in the description of the Julia set J from Theorem 1.2 we need to consider $\lambda = e^{2\pi ip/q}$, different from 1. To do so, we have generalized a theorem of Ueda [U] and Hakim [Ha] regarding the local normal form around the semi-parabolic fixed point from the case $\lambda = 1$ to the case $\lambda = e^{2\pi ip/q}$. This is given in Section 3. In Section 4 we define big attractive petals for semi-parabolic germs of $(\mathbb{C}^2, 0)$. Both sections are of independent interest. In Theorem 6.2 we show how to control the size of the normalizing neighborhood for our family of semi-parabolic Hénon maps.

Remark 1.5. We were able to characterize J without using J^- , by carefully describing the set J^+ inside the polydisk $\mathbb{D}_r \times \mathbb{D}_r$. We know from [BS8] that the Hénon map is hyperbolic on J if and only if there is a neighborhood \mathcal{N} of J and Riemann surface laminations \mathcal{L}^\pm of $\mathcal{N} \cap J^\pm$ such that \mathcal{L}^+ and \mathcal{L}^- intersect transversely at all points of J . In our setting, we have shown that J^+ is laminar and $J = J^*$, but the Hénon map is semi-parabolic (hence not hyperbolic), so J^- is non-laminar or J^+ and J^- may have points of non-transverse intersection. From Theorem 1.2, J is connected and by Theorem 1.5 in [Du] it follows that the set $J^- - K^+$ supports a unique Riemann surface lamination which is uniquely ergodic. In fact, it seems reasonable that J^- is non-laminar precisely at the semi-parabolic fixed point.

Remark 1.6. Let $(H_a)_{a \in \mathbb{D}_\delta}$ be the family of complex Hénon maps with a semi-parabolic fixed point with one eigenvalue $\lambda = e^{2\pi ip/q}$ from Theorem 1.1. It follows from Bedford, Lyubich and Smillie [BLS] that H_a admits an invariant measure μ_a which is the unique measure of maximal entropy $\log(2)$. The measure μ_a has two non-zero Lyapunov exponents $\lambda_a^- < 0 < \lambda_a^+$. Let J_a denote the Julia set of H_a . We have the following dichotomy from [BS5]: $\lambda_a^- = \log(2)$ if and only if J_a is connected. We have shown in Theorem 1.2, that the Julia set J_a is connected for each $a \in \mathbb{D}_\delta$. Thus $\lambda_a^+ = \log(2)$ and $\lambda_a^- = 2 \log |a| - \log(2)$ for this family of semi-parabolic Hénon maps.

In Theorem 1.1 we give a characterization of semi-parabolic Hénon maps $H_{c,a}$ that are perturbations of a parabolic polynomial $p(x) = x^2 + c_0$. This generalizes the theorem of Hubbard and Oberste-Vorth [HOV2], which describes Hénon maps that are perturbations of a hyperbolic polynomial, to the semi-parabolic setting. The technique of our proof is quite new and is inspired by the proof of Douady and Hubbard [DH], Section X, (see also [H1]) that the Julia set of a parabolic polynomial is locally connected. They show that the (inverse) Böttcher isomorphism extends continuously to the boundary for quadratic polynomials with a parabolic cycle, thus showing local connectivity of the Julia set. We create a two-dimensional analogue to show connectivity of the Julia set for the semi-parabolic Hénon map.

The key to proving Theorem 1.1 is to build a metric on a neighborhood of $J^+ \cap V$ for which the Hénon map is expanding in the horizontal direction. We will consider the infimum of a pull-back of an Euclidean metric in a small tubular neighborhood of the local stable manifold of the semi-parabolic fixed point and a product of a Poincaré metric with an Euclidean metric outside. The details are given in Section 8. The expanding factor depends on the distance to the local stable manifold of the semi-parabolic fixed point, so it is strictly bigger than one, but there is no constant of uniform expansion.

We prove the result in Theorem 1.1 as a Browder fixed point theorem [Br]. We will recover the set J^+ inside a bidisk as the image of the unique fixed point of a weakly contracting graph-transform operator in an appropriate function space \mathcal{F} . The space \mathcal{F} and the contraction are described in Sections 10 and 11. In order to establish the conjugacy of the semi-parabolic Hénon map to the model map ψ we have used some heavy-duty topology: a theorem of Hamstrom [Ham], which states that if S is a compact surface with nonempty boundary then the components of the group of homeomorphisms which are the identity on the boundary are contractible. This is described in detail in Section 12. The approach outlined above was used in [R] to reprove the theorem of Hubbard and Oberste-Vorth about hyperbolic Hénon maps [HOV2] as an application of the Banach fixed point theorem.

This article is built on previous work done by the authors in [R] and [T]. We will further use the techniques developed in this paper in [RT] to study perturbations of semi-parabolic Hénon maps. We show in [RT] that the family of semi-parabolic Hénon maps H_a , where a belongs to an open disk of parameters $|a| < \delta$ from \mathcal{P}_λ , lies in the boundary of a hyperbolic component of the Hénon connectedness locus.

Remark 1.7. For the family of semi-parabolic Hénon maps with small enough Jacobian (suppose $|a| < \delta$ as in Theorem 1.1) there are no wandering components of $\text{int}(K^+)$. The proof is given in [R] and is similar to the hyperbolic case from [BS2]. This essentially follows from the fact that the Hénon map expands in horizontal cones on a neighborhood of $J^+ \cap V$ as shown in Sections 8 and 9.

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2. PRELIMINARIES

For a polynomial p of degree $d \geq 2$, the *filled Julia set* of p is

$$K_p = \{z \in \mathbb{C} \mid |p^{on}(z)| \text{ bounded as } n \rightarrow \infty\}.$$

The set $J_p = \partial K_p$ is the *Julia set* of p . If K_p is connected (or equivalently J_p is connected) then there exists a unique analytic isomorphism

$$\Phi_p : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - K_p \tag{1}$$

such that $\Phi_p(z^d) = p(\Phi_p(z))$ and $\Phi_p(z)/z \rightarrow 1$ as $z \rightarrow \infty$. Furthermore, if J_p is locally connected then Φ_p extends to the boundary \mathbb{S}^1 and defines a continuous, surjective map $\gamma : \mathbb{S}^1 \rightarrow J_p$ [M]. The Julia set of a hyperbolic or parabolic polynomial is locally connected [DH]. The boundary map γ is called the Carathéodory loop. We will use this map in an essential way in Section 12.

Fix $\lambda = e^{2\pi i p/q}$ a root of unity. The set of parameters $(c, a) \in \mathbb{C}^2$ for which the Hénon map $H_{c,a}$ has a fixed point with one eigenvalue λ is a curve of equation

$$\mathcal{P}_\lambda := \left\{ (c, a) \in \mathbb{C}^2 \mid c = c(a) := (1 - a^2) \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda} \right) - \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda} \right)^2 \right\}. \tag{2}$$

To see this, let (x, y) be a fixed point of the Hénon map such that the derivative $DH_{c,a}$ at (x, y) has an eigenvalue λ . Then λ is a root of the characteristic polynomial $\lambda^2 - 2x\lambda - a^2 = 0$. The parameters (c, a) must verify the equations $x^2 + c + ay = x$, $y = ax$ and $x = \frac{\lambda}{2} - \frac{a^2}{2\lambda}$. The solution set is the curve \mathcal{P}_λ .

The parameter c is a quadratic function of the Jacobian $-a^2$, so we will refer to the curves \mathcal{P}_λ as parabolas. For (c, a) in \mathcal{P}_λ , $H_{c,a}$ has a fixed point \mathbf{q}_a such that $DH_{c,a}(\mathbf{q}_a)$ has one eigenvalue λ and one eigenvalue $\mu = -\frac{a^2}{\lambda}$. When $|a| < 1$, the eigenvalue μ is smaller than one in absolute value, and we call \mathbf{q}_a a *semi-parabolic fixed point*, and $H_{c,a}$ a *semi-parabolic Hénon map*. The fixed point has an explicit equation

$$\mathbf{q}_a := \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}, a \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda} \right) \right). \quad (3)$$

We will use this notation throughout this paper. We will see that for δ small enough and $(c, a) \in \mathcal{P}_\lambda$ with $0 < |a| < \delta$, the semi-parabolic fixed point \mathbf{q}_a has multiplicity $q+1$ as a solution of the equation $H_a^{cq}(x, y) = (x, y)$. In analogy with the one dimensional dynamics, we say that the semi-parabolic multiplicity of \mathbf{q}_a in this case is 1.

The semi-parabolic fixed point \mathbf{q}_a has a strong stable manifold $W^s(\mathbf{q}_a)$ biholomorphic to \mathbb{C} corresponding to the eigenvalue μ whose absolute value is strictly less than 1,

$$W^s(\mathbf{q}_a) := \{\mathbf{p} \in \mathbb{C}^2 \mid \|H^{om}(\mathbf{p}) - \mathbf{q}_a\| < C|\mu|^m \text{ for } m \geq 0\}, \quad (4)$$

where $C > 0$ is a constant [U]. This is the set of points for which $H^{om}(\mathbf{p}) \rightarrow \mathbf{q}_a$ exponentially as $m \rightarrow \infty$. Bedford, Smillie and Ueda show that $W^s(\mathbf{q}_a)$ is dense in J^+ [BSU]. The basin of attraction of the semi-parabolic fixed point \mathbf{q}_a belongs to $\text{int}(K^+)$ and is a Fatou-Bieberbach domain [Ha], [U]. The rate of convergence to \mathbf{q}_a is parabolic.

The parametric line $a = 0$ intersects the curve \mathcal{P}_λ at the point $c_0 = \frac{\lambda}{2} - \frac{\lambda^2}{4}$. It is easy to see that the polynomial $p(x) = x^2 + c_0$ has a parabolic fixed point $q_0 = \frac{\lambda}{2}$ of multiplier λ . This is the polynomial from which we are perturbing in \mathbb{C}^2 . Let J_p and K_p denote the Julia set, respectively the filled-in Julia set of the polynomial p .

For $(c, a) \in \mathcal{P}_\lambda$ the equation for c from 2 can be rewritten as

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4} + a^2w, \quad \text{where } w := \frac{2\lambda - 2\lambda^2 - 1}{4\lambda} + a^2 \frac{2\lambda - 1}{4\lambda^2}. \quad (5)$$

Thus we can also write the semi-parabolic Hénon map $H_{c,a}(x, y) = (x^2 + c + ay, ax)$ as

$$H_a(x, y) = (p(x) + a^2w + ay, ax), \quad (6)$$

with inverse

$$H_a^{-1}(x, y) = \frac{1}{a} (y, x - p(y/a) - a^2w), \quad (7)$$

where p is the parabolic polynomial $p(x) = x^2 + c_0$. This emphasizes the dependency on the parabolic polynomial p . The constant w depends only on a and λ and clearly $|w| < 2$.

Following [HOV1], we choose a constant r greater than the largest root of the quadratic equation $|x|^2 - (|a| + 2)|x| - |c_0| - |a|^2|w| = 0$. Then the dynamical space \mathbb{C}^2 can be divided into three regions: the bidisk $\mathbb{D}_r \times \mathbb{D}_r = \{(x, y) \in \mathbb{C}^2 \mid |x| \leq r, |y| \leq r\}$,

$$V^+ = \{(x, y) \mid |x| \geq \max(|y|, r)\} \quad \text{and} \quad V^- = \{(x, y) \mid |y| \geq \max(|x|, r)\}.$$

The sets J and K are contained in $\mathbb{D}_r \times \mathbb{D}_r$. The escaping sets are $U^+ = \mathbb{C}^2 - K^+$ and $U^- = \mathbb{C}^2 - K^-$ and they can be described in terms of V^+ and V^- as follows [HOV1]:

$$U^+ = \bigcup_{k \geq 0} H^{-\circ k}(V^+) \quad \text{and} \quad U^- = \bigcup_{k \geq 0} H^{\circ k}(V^-).$$

Therefore the Julia set J^+ is the common boundary of K^+ and U^+ . We know that $|c_0| < 2$ from [DH] so any constant $r > 3$ works for our purpose.

In order to understand these objects better it is useful to look first at the case when $a = 0$. In this case the dynamics of the Hénon map reduces to the dynamics of the polynomial p . The relevant sets under forward dynamics can then be easily described: $J^+ = J_p \times \mathbb{C}$, $K^+ = K_p \times \mathbb{C}$ and $U^+ = (\mathbb{C} - K_p) \times \mathbb{C}$.

3. NORMAL FORM OF SEMI-PARABOLIC HÉNON MAPS

Hakim in [Ha] and Ueda in [U] have studied normal forms for germs of semi-attractive transformations H of $(\mathbb{C}^n, 0)$ for which $DH_{(0)}$ has one eigenvalue $\lambda = 1$, and the other eigenvalues μ_2, \dots, μ_n have absolute values $|\mu_j| < 1, j = 2, \dots, n$.

The following results are similar to Proposition 2.1, 2.2, and 2.3 from [Ha] and to Section 6 from [U]. We have adapted the propositions in [Ha] to semi-parabolic germs of transformations of $(\mathbb{C}^2, 0)$ with eigenvalues $\lambda = e^{2\pi i p/q}$ and $|\mu| < 1$. As a consequence we get that 0 is a fixed point with multiplicity $\nu q + 1$ for some constant ν which we call the *(semi) parabolic multiplicity* of the fixed point, like in one-dimensional dynamics.

Proposition 3.1. *Let H be a semi-parabolic germ of transformation of $(\mathbb{C}^2, 0)$, with eigenvalues λ and μ , with $\lambda = e^{2\pi i p/q}$ and $|\mu| < 1$. There exist local coordinates (x, y) in which H has the form $H(x, y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = a_1(y)x + a_2(y)x^2 + \dots \\ y_1 = \mu y + xh(x, y) \end{cases} \quad (8)$$

where $a_j(\cdot)$ and $h(\cdot, \cdot)$ are germs of holomorphic functions from $(\mathbb{C}, 0)$ to \mathbb{C} , respectively from $(\mathbb{C}^2, 0)$ to \mathbb{C} , with $a_1(0) = \lambda$ and $h(0, 0) = 0$.

Proof. The proof is the same as in [Ha] and [U] and is based on the straightening of the local strong stable manifold of the fixed point. \square

Proposition 3.2. *Let H be a semi-parabolic germ of transformation of $(\mathbb{C}^2, 0)$, with eigenvalues λ and μ , with $\lambda = e^{2\pi i p/q}$ and $|\mu| < 1$. For every integer m there exist local coordinates (x, y) in which H has the form $H(x, y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = \lambda x + a_2 x^2 + \dots + a_m x^m + a_{m+1}(y)x^{m+1} + \dots \\ y_1 = \mu y + xh(x, y) \end{cases} \quad (9)$$

where a_2, \dots, a_m constants.

Proof. The proof is the same as in Proposition 2.2 from [Ha] (proved also in Section 6 of [U]). We will refer to this proof when we discuss the domain of convergence of the

functions $u(\cdot)$ and $v(\cdot)$ defined below. We know from Proposition 3.1 that there exist local coordinates (x, y) in which H has the form

$$\begin{cases} x_1 = a_1(y)x + a_2(y)x^2 + \dots \\ y_1 = \mu y + xh(x, y). \end{cases}$$

The germs $a_i(\cdot)$ and $h(\cdot, \cdot)$ germs of holomorphic functions from $(\mathbb{C}, 0)$ to \mathbb{C} , respectively from $(\mathbb{C}^2, 0)$ to \mathbb{C} , with $a_1(0) = \lambda$ and $h(0, 0) = 0$.

(1) Reduction to $a_1(y) = \lambda$. Consider as in [Ha] and [U] a coordinate transformation

$$\begin{cases} X = u(y)x \\ Y = y \end{cases} \quad \text{with inverse} \quad \begin{cases} x = X/u(Y) \\ y = Y \end{cases}$$

where u is a germ of analytic functions from $(\mathbb{C}, 0)$ to \mathbb{C} with $u(0) = \lambda$. We need to find u such that

$$\begin{aligned} X_1 &= u(y_1)x_1 = u(\mu y + xh(x, y)) (a_1(y)x + a_2(y)x^2 + \dots) \\ &= u(\mu Y + X/u(Y)h(X/u(Y), Y)) (a_1(Y)X/u(Y) + a_2(Y)(X/u(Y))^2 + \dots) \\ &= \frac{u(\mu Y)a_1(Y)}{u(Y)}X + \mathcal{O}(X^2) = \lambda X + \mathcal{O}(X^2). \end{aligned}$$

Thus u satisfies the equation $u(Y) = u(\mu Y) \frac{a_1(Y)}{\lambda}$. We successively substitute μY instead of Y in this equation and obtain the unique solution

$$u(Y) = \prod_{n=0}^{\infty} \frac{a_1(\mu^n Y)}{\lambda}. \quad (10)$$

This series converges in a neighborhood of 0 since $\mu < 1$ and $a_1(0) = \lambda$.

(2) Reduction to $a_2(y), \dots, a_m(y)$ constants. We proceed by induction on m . The base case $m = 1$ was discussed above. Suppose that $m \geq 2$ and that there exist local coordinates (x, y) in which H has the form

$$\begin{cases} x_1 = \lambda x + a_2 x^2 + \dots + a_{m-1} x^{m-1} + a_m(y) x^m + \dots \\ y_1 = \mu y + xh(x, y), \end{cases}$$

with a_2, \dots, a_{m-1} constant. We would like to find local coordinates so that $a_m(y)$ is also constant. Consider the transformation

$$\begin{cases} X = x + v(y)x^m \\ Y = y \end{cases} \quad \text{with inverse} \quad \begin{cases} x = X - v(Y)X^m + \dots \\ y = Y \end{cases}$$

where v is a germ of analytic functions from $(\mathbb{C}, 0)$ to \mathbb{C} with $v(0) = 0$. Using the coordinates given by this transformation we get

$$\begin{aligned} X_1 &= x_1 + v(y_1)x_1^m \\ &= \lambda x + a_2 x^2 + \dots + a_{m-1} x^{m-1} + (a_m(y) + v(\mu y)) x^m + \mathcal{O}(x^{m+1}) \\ &= X - v(Y)X^m + a_2 X^2 + \dots + a_{m-1} X^{m-1} + (a_m(Y) + v(\mu Y)) X^m + \mathcal{O}(X^{m+1}) \\ &= X + a_2 X^2 + \dots + a_{m-1} X^{m-1} + (a_m(Y) + v(\mu Y) - v(Y)) X^m + \mathcal{O}(X^{m+1}). \end{aligned}$$

We need v such that the coefficient of X^m is constant, i.e. $a_m(Y) + v(\mu Y) - v(Y) = a_m(0)$ is constant. This gives the equation $v(Y) - v(\mu Y) = a_m(Y) - a_m(0)$. We successively

substitute μY instead of Y in this equation and obtain

$$v(Y) = \sum_{n=0}^{\infty} (a_n(\mu^n Y) - a_n(0)). \quad (11)$$

The series clearly converges in a neighborhood of 0 since $\mu < 1$. \square

Proposition 3.3. *Let H be a semi-parabolic germ of transformation of $(\mathbb{C}^2, 0)$, with eigenvalues λ and μ , with $\lambda = e^{2\pi i p/q}$ and $|\mu| < 1$. There exist local coordinates (x, y) in which H has the form $H(x, y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = \lambda(x + x^{\nu q+1} + Cx^{2\nu q+1} + a_{2\nu q+2}(y)x^{2\nu q+2} + \dots) \\ y_1 = \mu y + xh(x, y) \end{cases} \quad (12)$$

and C a constant. Moreover the multiplicity of the fixed point is $\nu q + 1$.

Proof. Suppose that the map has the form from Equation 9, where m is big enough, and fixed

$$\begin{cases} x_1 = \lambda x + a_k x^k + \dots + a_m x^m + a_{m+1}(y)x^{m+1} + \dots \\ y_1 = \mu y + xh(x, y). \end{cases}$$

Consider the coordinate transformation

$$\begin{cases} X = x + bx^k \\ Y = y \end{cases} \quad \text{with inverse} \quad \begin{cases} x = X - bX^k + \dots \\ y = Y \end{cases}$$

In the new coordinate system, we get

$$\begin{aligned} X_1 &= x_1 + bx_1^k = (\lambda x + a_k x^k + \dots) + b(\lambda x + a_k x^k + \dots)^k \\ &= \lambda x + a_k x^k + \dots + b\lambda^k x^k + \dots \\ &= \lambda x + (a_k + b\lambda^k)x^k + \dots \\ &= \lambda(X - bX^k + \dots) + (a_k + b\lambda^k)(X - bX^k + \dots)^k + \dots \\ &= \lambda X + (a_k + b(\lambda^k - \lambda))X^k + \dots \end{aligned}$$

If k is not congruent to 1 modulo q (i.e. $\lambda^k \neq \lambda$), then we can set

$$b = \frac{a_k}{\lambda - \lambda^k}$$

and eliminate the term $a_k x^k$. This proves that by successive coordinate transformations of the form $X = x + bx^k, Y = y$ we can eliminate terms with powers that are not congruent to 1 modulo q , so the first term that cannot be eliminated in this way will have a power of the form $\nu q + 1$ for some ν .

Thus the map takes the form

$$\begin{cases} x_1 = \lambda(x + a_{\nu q+1}x^{\nu q+1} + \dots + a_m x^m + a_{m+1}(y)x^{m+1} + \dots) \\ y_1 = \mu y + xh(x, y). \end{cases} \quad (13)$$

Here we will assume that m was chosen so that $m > 2\nu q + 1$. Thus the coefficients up to order m are still constants. We can further reduce Equation 13 to $a_{\nu q+1} = 1$ by

considering a transformation of the form $X = Ax, Y = y$, where A is a constant such that $A^{\nu q} = a_{\nu q+1}$. Consider therefore the transformation H written as

$$\begin{cases} x_1 = \lambda(x + x^{\nu q+1} + \dots + a_m x^m + a_{m+1}(y)x^{m+1} + \dots) \\ y_1 = \mu y + xh(x, y). \end{cases} \quad (14)$$

We have previously showed that we can eliminate any term of degree k between 1 and m , which is not congruent to 1 modulo q . One can also eliminate all terms of degree $j q + 1$, where $\nu < j < 2\nu$. Assume a_{jq+1} is the first such coefficient different from 0 as below

$$\begin{cases} x_1 = \lambda(x + x^{\nu q+1} + a_{jq+1}x^{jq+1} + \dots) \\ y_1 = \mu y + xh(x, y). \end{cases} \quad (15)$$

Consider the transformation

$$\phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{where} \quad \begin{cases} X = x + bx^{(j-\nu)q+1} \\ Y = y \end{cases} \quad \text{and} \quad b = \frac{a_{jq+1}}{(2\nu - j)q}.$$

Define $G := \phi \circ H \circ \phi^{-1}$ and suppose that $G(X, Y) = (X_1, Y_1)$ with

$$\begin{cases} X_1 = \lambda(X + X^{\nu q+1} + AX^k + \dots) \\ Y_1 = \mu Y + Xh(X, Y) \end{cases} \quad (16)$$

and $k \leq jq$. We show that $A = 0$ by comparing the terms of the power series of $G \circ \phi$ and $\phi \circ H$. We will only need to analyze the x -coordinate. The first coordinate of $\phi \circ H$ is

$$\begin{aligned} X_1 &= x_1 + bx_1^{(j-\nu)q+1} \\ &= \lambda(x + x^{\nu q+1} + a_{jq+1}x^{jq+1} + \dots) + b\lambda(x + x^{\nu q+1} + a_{jq+1}x^{jq+1} + \dots)^{(j-\nu)q+1} \\ &= \lambda \left(x + bx^{(j-\nu)q+1} + x^{\nu q+1} + (a_{jq+1} + b((j-\nu)q+1))x^{jq+1} + \mathcal{O}_y(x^{jq+2}) \right). \end{aligned}$$

The first coordinate of $G \circ \phi$ is

$$\begin{aligned} X_1 &= \lambda(X + X^{\nu q+1} + AX^k + \dots) \\ &= \lambda \left((x + bx^{(j-\nu)q+1}) + (x + bx^{(j-\nu)q+1})^{\nu q+1} + A(x + bx^{(j-\nu)q+1})^k + \dots \right) \\ &= \lambda \left(x + bx^{(j-\nu)q+1} + x^{\nu q+1} + b(\nu q + 1)x^{jq+1} + Ax^k + \mathcal{O}_y(x^{k+1}) \right). \end{aligned}$$

We have that $a_{jq+1} + b((j-\nu)q+1) = b(\nu q + 1)$ by the choice for b . The two power series are equal so the coefficient of x^k vanishes, so $A = 0$. Thus in the first coordinate of $\phi \circ H \circ \phi^{-1}$ the coefficient of x^{jq+1} is zero and the coordinate transformation did not introduce additional terms of lower powers.

Using similar transformations we can eliminate all terms between $\nu q + 1$ and $2\nu q + 1$ and write $H(x, y) = (x_1, y_1)$ with

$$\begin{cases} x_1 = \lambda(x + x^{\nu q+1} + Cx^{2\nu q+1} + \mathcal{O}_y(x^{2\nu q+2})) \\ y_1 = \mu y + xh(x, y) \end{cases}$$

for some constant C .

It is easy to prove that in the last coordinate system, $H^{\circ q}$ takes the form

$$\begin{cases} x_1 = x + qx^{\nu q+1} + \tilde{C}x^{2\nu q+1} + \mathcal{O}_y(x^{2\nu q+2}) \\ y_1 = \mu^q y + x\tilde{h}(x, y). \end{cases} \quad (17)$$

The partial derivative $\frac{\partial y_1}{\partial y}(0,0) = \mu^q < 1$, hence by the Implicit Function Theorem, the equation $\mu^q y + xh(x,y) = y$ has a unique solution $y = \varphi(x)$ in a neighborhood of 0, where φ is a holomorphic function. From the first equation it then follows that $x = 0$ is a fixed point of $H^{\circ q}$ of multiplicity $\nu q + 1$. \square

The normalizing form as proven in the previous theorem holds locally around the semi-parabolic fixed point. The disadvantage of the “local” statement is that it does not allow us to control the size of the neighborhood of the fixed point where we can put on normalizing coordinates. However, in Section 6 we show how to control the size of this neighborhood. We consider a class of semi-parabolic Hénon maps which are perturbations of a polynomial with a parabolic fixed point or cycle, and show how to extend this theorem in order to get uniform bounds (with respect to the parameters) on the size of the normalizing neighborhood.

4. ATTRACTING AND REPELLING SECTORS

Set $m := \nu q$ and let

$$\Delta_R = \left\{ x \in \mathbb{C} \mid \left(\operatorname{Re}(x^m) + \frac{1}{2R} \right)^2 + \left(\operatorname{Im}(x^m) - \frac{1}{2R} \right)^2 < \frac{1}{2R^2} \right\}.$$

There are m connected components of Δ_R , which we denote $\Delta_{R,j}$, for $1 \leq j \leq m$. Define

$$\mathcal{P}_{att} = \{(x, y) \in \mathbb{C}^2 \mid x \in \Delta_R, |y| < r\}$$

and let

$$\mathcal{P}_{att,j} = \{(x, y) \in \mathbb{C}^2 \mid x \in \Delta_{R,j}, |y| < r\}$$

be the connected components of \mathcal{P}_{att} . These are called *(big) attractive petals* for the Hénon map, similar to the one-dimensional case.

Proposition 4.1. *For R large enough and r small enough*

$$H(\overline{\mathcal{P}_{att,j}}) \subset \mathcal{P}_{att,j+\nu p} \cup \{0\} \times \mathbb{D}_r \quad \text{for } 1 \leq j \leq \nu q.$$

In particular $H(\overline{\mathcal{P}_{att}}) \subset \mathcal{P}_{att} \cup \{0\} \times \mathbb{D}_r$ and all points of \mathcal{P}_{att} are attracted to the origin under iterations by H .

Proof. The analysis is similar to [Ha], but one should have in mind the formalism from the one-dimensional case (see [DH], [BH]) to resolve the ambiguity about which branch of $x^{1/m}$ we are talking about.

Assume that R is large enough and r is small enough so that H is well defined and has the expansion from Proposition 3.3. Define the region U_{R_1}

$$U_{R_1} := \{X \in \mathbb{C} \mid R_1 - \operatorname{Re}(X) < |\operatorname{Im}(X)|\}$$

where $R_1 = R/m$ and set $W_{R_1,r} := U_{R_1} \times \mathbb{D}_r \subset \mathbb{C}^2$.

Consider the Hénon map H written as

$$\begin{cases} x_1 = \lambda(x + x^{m+1} + Cx^{2m+1} + a_{2m+2}(y)x^{2m+2} + \dots) \\ y_1 = \mu y + xh(x, y). \end{cases}$$

Suppose $(x, y) \in \mathcal{P}_{att,j}$ and consider the transformation

$$\begin{cases} X = -\frac{1}{mx^m} \\ Y = y. \end{cases}$$

It maps each $\mathcal{P}_{att,j}$ to $W_{R_1,r}$ (it maps points $(0, y)$ to (∞, y)). Let $\hat{H}(X, Y) = (X_1, Y_1)$ be the map in these coordinates

$$\begin{aligned} X_1 &= -\frac{1}{mx_1^m} = -\frac{1}{m(\lambda(x + x^{m+1} + Cx^{2m+1} + a_{2m+2}(y)x^{2m+2} + \dots))^m} \\ &= \frac{X}{(1 + x^m + Cx^{2m} + a_{2m+2}(y)x^{2m+1} + \dots)^m} \\ &= X \left(1 - m(x^m + Cx^{2m} + \dots) + \frac{m(m+1)}{2}x^{2m} + \dots \right) \\ &= X + 1 + \frac{A}{X} + \mathcal{O}_Y \left(\frac{1}{|X|^{1+1/m}} \right) \end{aligned}$$

where $A := \frac{1}{m} \left(\frac{m+1}{2} - C \right)$ is a constant. The notation $\mathcal{O}_Y \left(\frac{1}{|X|^\alpha} \right)$ represents a holomorphic function of (X, Y) in $W_{R_1,r}$ which is bounded by $\frac{K}{|X|^\alpha}$ for some constant K . Similarly

$$Y_1 = \mu y + xh(x, y) = \mu Y + \mathcal{O}_Y \left(\frac{1}{|X|^{1/m}} \right).$$

Note that $|X| > \frac{R_1}{\sqrt{2}}$ for all $X \in U_{R_1}$. There exists constants K' and K'' such that

$$\begin{aligned} |X_1 - X - 1| &\leq \frac{K'}{|X|} < \frac{K_1}{R_1} \quad \text{where } K_1 := K'\sqrt{2} \\ |Y_1 - \mu Y| &\leq \frac{K''}{|X|^{1/m}} < \frac{K_2}{R_1^{1/m}} \quad \text{where } K_2 := K''\sqrt{2}^{1/m}. \end{aligned}$$

Choose R_1 large enough and r small enough so that

$$\begin{cases} \frac{K_1}{R_1} < \frac{1}{2} \\ \frac{K_2}{R_1^{1/m}} < (1 - |\mu|)r. \end{cases} \quad (18)$$

The first condition gives $|X_1 - X - 1| < \frac{1}{2}$, which implies $Re(X_1) > Re(X) + \frac{1}{2}$ and $|Im(X_1)| > |Im(X)| - \frac{1}{2}$. Thus $R_1 - Re(X_1) < |Im(X_1)|$. The second condition gives

$$|Y_1| \leq |Y_1 - \mu Y| + |\mu||Y| < \frac{K_2}{R_1^{1/m}} + |\mu|r < r.$$

Hence $\hat{H}(W_{R_1,r}) \subset W_{R_1,r}$.

We need to show that points in $W_{R_1,r}$ are attracted by $(\infty, 0)$ under iterations by \hat{H} . Let $(X, Y) \in W_{R_1,r}$ and set $(X_n, Y_n) = \hat{H}^{\circ n}(X, Y)$. Assume without loss of generality that $Re(X) > \rho$, where $\rho > 0$ is a constant to be defined later. We can make this assumption since $Re(X_k) > Re(X) + \frac{k}{2}$ for every positive integer k . We take the first integer k_0 such that $Re(X_{k_0}) > \rho$ and let $X := X_{k_0}$ and $Y := Y_{k_0}$.

Clearly

$$\operatorname{Re}(X_n) > \rho + \frac{n}{2} \quad (19)$$

for every $n \geq 0$. This follows immediately by induction since

$$\operatorname{Re}(X_{n+1}) > \operatorname{Re}(X_n) + 1/2 > \rho + (n+1)/2.$$

We now show by induction that

$$|Y_n| < 2NrR_1^{1/m} \left(\frac{1}{\rho + \frac{n}{2}} \right)^{1/m}, \quad n \geq 0$$

where N is an integer number such that $NR_1^{1/m} > \rho^{1/m}$. When $n = 0$, $|Y| < r$ and

$$r < 2NrR_1^{1/m} \frac{1}{\rho^{1/m}} \Leftrightarrow \rho^{1/m} < 2NR_1^{1/m}.$$

We now proceed to the induction step. First note that $|X_n| \geq \operatorname{Re}(X_n) > \rho + \frac{n}{2}$ and $K'' < K_2$. We get

$$\begin{aligned} |Y_{n+1}| &\leq |Y_{n+1} - \mu Y_n| + |\mu| |Y_n| < \frac{K''}{|X_n|^{1/m}} + |\mu| |Y_n| \\ &< (K_2 + |\mu| 2NrR_1^{1/m}) \left(\frac{1}{\rho + \frac{n}{2}} \right)^{1/m} < (1 + (2N-1)|\mu|) r R_1^{1/m} \left(\frac{1}{\rho + \frac{n}{2}} \right)^{1/m} \end{aligned}$$

and we want to show that

$$|Y_{n+1}| < 2NrR_1^{1/m} \left(\frac{1}{\rho + \frac{n+1}{2}} \right)^{1/m}.$$

This inequality is satisfied if

$$\left(\frac{\rho + \frac{1}{2}}{\rho} \right)^{1/m} = \left(1 + \frac{1}{2\rho} \right)^{1/m} < \frac{2}{1 + |\mu|} \leq \frac{2N}{1 + (2N-1)|\mu|}. \quad (20)$$

But $|\mu| < 1$, so $2/(1 + |\mu|) > 1$. This allows us to choose a number ρ large enough so that Equation 20 is satisfied. Then choose an integer N such that $NR_1^{1/m} > \rho^{1/m}$.

It follows that $(X_n, Y_n) \rightarrow (\infty, 0)$ as $n \rightarrow \infty$. \square

Let $\epsilon_0 = \tan(\pi/12)$. Define the *attractive sectors*

$$\Delta^+ := \left\{ x \in \mathbb{C} \mid \operatorname{Re}(x^m) \leq \epsilon_0 |\operatorname{Im}(x^m)| \text{ and } |x^m| < \frac{1}{\sqrt{2}R} \right\} \quad (21)$$

and *repelling sectors*

$$\Delta^- := \left\{ x \in \mathbb{C} \mid \operatorname{Re}(x^m) > \epsilon_0 |\operatorname{Im}(x^m)| \text{ and } |x^m| < \frac{1}{\sqrt{2}R} \right\}. \quad (22)$$

Let $W^+ = \Delta^+ \times \mathbb{D}_r \subset \mathcal{P}_{att} \cup \{0\} \times \mathbb{D}_r$ and $W^- = \Delta^- \times \mathbb{D}_r$. We call W^- repelling because as we will see, the Hénon map expands horizontally when the Jacobian is small enough. There are m components of W^\pm which we denote W_j^\pm for $1 \leq j \leq m$. These are the preimages of the red/green regions in Figure 2 under $x \mapsto x^m$. The choice of ϵ_0 means that the angle of the image of W^- under $x \mapsto x^m$ is $5\pi/6$.

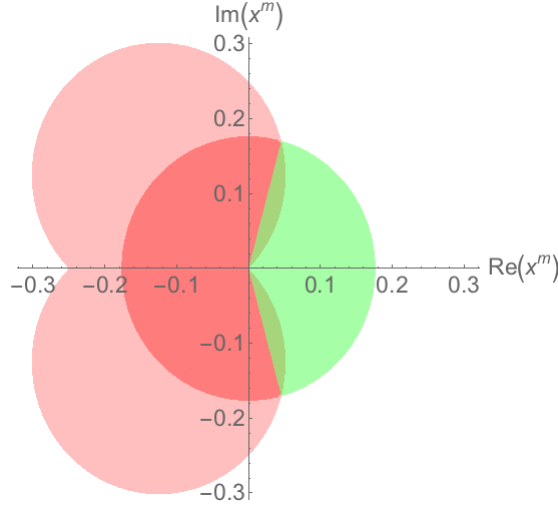


Figure 2. The image of \mathcal{P}_{att} under the map $x \mapsto x^m$ at height $y = 0$ is shown in light red. Similarly the attracting sector Δ^+ is shown in red and the repelling sector Δ^- in green. The angle opening of the green region is $5\pi/6$.

Furthermore, since $Re(x^m) > \epsilon_0 |Im(x^m)|$ on W^- , we have

$$Re(x^m) > \epsilon_1 |x^m|, \quad \text{where } \epsilon_1 := \frac{\epsilon_0}{\sqrt{1 + \epsilon_0^2}} > \frac{1}{4}. \quad (23)$$

Proposition 4.2. *The interior of the union $\bigcup_{n \geq 0} H^{-\circ n}(W^+)$ is the basin of attraction of the semi-parabolic fixed point.*

Proof. The proof follows immediately by analyzing the situation at infinity using Equation 19 as in the proof of Proposition 4.1. \square

5. THE PARAMETRIZING MAP OF THE STABLE MANIFOLD

This is a self-contained section where we study the degeneracy of the parametrization of the stable manifold of the semi-parabolic fixed point \mathbf{q}_a as $a \rightarrow 0$. Consider the Hénon map H and its inverse H^{-1} written as in Equation 6, respectively 7.

Fix $\lambda = e^{2\pi i p/q}$. Suppose H has a semi-parabolic fixed point at \mathbf{q}_a such that $DH(\mathbf{q}_a)$ has eigenvalues λ and μ , with $|\mu| < 1$. We have $\lambda\mu = -a^2$ so $\mu = -\frac{a^2}{\lambda}$ and $|\mu| = |a|^2$. Set for simplicity $q_a := \frac{\lambda}{2} - \frac{a^2}{2\lambda} = \frac{\lambda + \mu}{2}$. With this notation, the equation 3 of the fixed point \mathbf{q}_a reduces to

$$\mathbf{q}_a := \begin{pmatrix} q_a \\ a q_a \end{pmatrix} = \frac{\lambda + \mu}{2} \begin{pmatrix} 1 \\ a \end{pmatrix}.$$

Let $v = \begin{pmatrix} -a/\lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \mu/a \\ 1 \end{pmatrix}$ be an eigenvector for the eigenvalue μ . The semi-parabolic fixed point \mathbf{q}_a has a stable manifold $W^s(\mathbf{q}_a) \subset \mathbb{C}^2$ as in Equation 4. The stable manifold is biholomorphic to \mathbb{C} and has a natural parametrization given by the following proposition.

Proposition 5.1. *The stable manifold $W^s(\mathbf{q}_a)$ has a parametrization $F_a : \mathbb{C} \rightarrow W^s(\mathbf{q}_a)$ given by*

$$F_a(z) = \lim_{m \rightarrow \infty} H^{-\circ m}(\mathbf{q}_a + \mu^m v z). \quad (24)$$

F_a is an injective immersion of \mathbb{C} onto $W^s(\mathbf{q}_a)$ with the property that $F_a(\mu z) = H(F_a(z))$.

Proof. The proof is similar to the proof of Theorem 1 from [H2]. Consider the inverse map H^{-1} instead of H . Then \mathbf{q}_a is a fixed point of H^{-1} and $DH^{-1}(\mathbf{q}_a)$ has eigenvalues $\bar{\lambda}$ and $\mu' = 1/\mu$, where $|\mu'| > 1$. The fixed point \mathbf{q}_a has now an unstable manifold $W^u(\mathbf{q}_a)$ which has a natural parametrization given by F_a as shown in [H2]. \square

Proposition 5.2. *The parametrizing function $F_a \rightarrow F_0$ as $a \rightarrow 0$, where*

$$F_0(z) := \mathbf{q}_0 + (0, z) = (\lambda/2, z).$$

Proof. Define a sequence of points

$$(x_i, y_i) = H^{-1}(x_{i-1}, y_{i-1}) \quad \text{for } 1 \leq i \leq m,$$

where

$$(x_0, y_0) = \mathbf{q}_a + \mu^m v z = \left(q_a + \frac{\mu^{m+1}}{a} z, a q_a + \mu^m z \right).$$

At the first step we have

$$(x_1, y_1) = H^{-1}(x_0, y_0) = \frac{1}{a} (y_0, x_0 - p(y_0/a) - a^2 w),$$

so

$$x_1 = \frac{y_0}{a} = q_a + \frac{\mu^m}{a} z.$$

From the fixed point equation $H(\mathbf{q}_a) = \mathbf{q}_a$ we get that $p(q_a) + a^2 q_a + a^2 w = q_a$ so $q_a - p(q_a) - a^2 w = a^2 q_a$. Moreover, the matrix $DH(\mathbf{q}_a)$ has eigenvalues λ and μ so $\lambda + \mu = \text{tr}(DH(\mathbf{q}_a))$, which gives $p'(q_a) = \lambda + \mu$. Since $y_0 = a q_a + \mu^m z$ and $\mu = -\frac{a^2}{\lambda}$, we can write

$$p(y_0/a) = p\left(q_a + \frac{\mu^m}{a} z\right) = p(q_a) + p'(q_a) \frac{\mu^m}{a} z + \mathcal{O}(\mu^{2m-1}).$$

Note that this is a finite sum. Thus the equation for y_1 has the following form

$$\begin{aligned} y_1 = \frac{x_0 - p(y_0/a) - a^2 w}{a} &= \frac{q_a - p(q_a) - a^2 w}{a} + \frac{\mu^{m+1} z - p'(q_a) \mu^m z}{a^2} + \mathcal{O}(a \mu^{2m-2}) \\ &= a q_a + \mu^{m-1} z + \mathcal{O}(a \mu^{2m-2}). \end{aligned}$$

By induction we can show that for $1 \leq i \leq m$ we have

$$\begin{aligned} x_i &= q_a + \frac{\mu^{m-(i-1)}}{a} z + \mathcal{O}(\mu^{2(m-(i-1))}) \\ y_i &= a q_a + \mu^{m-i} z + \mathcal{O}(a \mu^{2(m-i)}). \end{aligned}$$

For $i = m$ these reduce to

$$\begin{aligned} x_m &= q_a - a \bar{\lambda} z + \mathcal{O}(a^4) \\ y_m &= a q_a + z + \mathcal{O}(a). \end{aligned}$$

Thus $x_m \rightarrow q_0$ and $y_m \rightarrow z$ as $a \rightarrow 0$. Therefore F_a converges to F_0 uniformly on compact subsets of \mathbb{C} as $a \rightarrow 0$. \square

6. CHOOSING UNIFORM NORMALIZING COORDINATES

In Section 3, we described the normal form of germs of transformations of \mathbb{C}^2 with a semi-parabolic fixed point. In this section, we will study the normal form for the family of Hénon maps with a semi-parabolic fixed point, which are small perturbations of the parabolic polynomial $p(x)$ inside the parabola \mathcal{P}_λ . Since \mathcal{P}_λ is parametrized by a , we write the Hénon map as $H_a(x, y) = (x^2 + c + ay, ax)$, where c is chosen as in Equation 2, so that $(c, a) \in \mathcal{P}_\lambda$.

We will show how to extend the results from Section 3 in order to get uniform bounds (with respect to the parameter a) on the size of the normalizing neighborhood. We will prove that the coordinate transformation ϕ_a that puts the Hénon map H_a in the normal form is holomorphic with respect to a . Then we will use the theory developed in Section 4 to exhibit local attractive and repelling sectors for the Hénon map H_a . The attractive sectors will belong to the interior of K^+ . In the repelling sectors we will show that the derivative of the Hénon map is weakly expanding in the “horizontal” direction and strongly contracting in the “vertical” direction.

We first look at the normal form from [DH] and [H1] for the polynomial $p(x) = x^2 + c_0$ which has a parabolic fixed point $q_0 = \frac{\lambda}{2}$, of multiplier $\lambda = e^{2\pi i p/q}$. Denote, for the clarity of exposition, $\rho := (\sqrt{2}R)^{-1/q}$ in the definition of the set Δ^\pm from Equations 21 and 22.

Lemma 6.1. *There exists a neighborhood V_0 of q_0 and an isomorphism $\phi : V_0 \rightarrow \mathbb{D}_\rho$ such that $\tilde{p}(x) = \phi \circ p \circ \phi^{-1}(x)$ where $\tilde{p}(x) = \lambda x (1 + x^q + Cx^{2q} + \mathcal{O}(x^{2q+1}))$. Furthermore, there exists ρ small enough such that in the region*

$$\Delta^- = \{|x| < \rho \mid \operatorname{Re}(x^q) > \epsilon_0 |\operatorname{Im}(x^q)|\}$$

the map \tilde{p} satisfies $|\tilde{p}'(x)| > 1 + \epsilon_1 |x|^q$. The compact region

$$\Delta^+ = \{|x| < \rho \mid \operatorname{Re}(x^q) \leq \epsilon_0 |\operatorname{Im}(x^q)|\}$$

satisfies $p(\Delta^+) \subset \operatorname{int}(K_p) \cup \{0\}$.

Proof. After a global coordinate change that brings the parabolic fixed point at the origin, we can write the polynomial as $p(x) = \lambda x + x^2$. Since p is a quadratic polynomial, the fixed point q_0 can only have parabolic multiplicity 1, hence its multiplicity as a solution of the equation $p^{\circ q}(z) - z = 0$ is $q + 1$. The local normal form of p around q_0 is

obtained by successive elimination of the terms of degree less than $2q+1$ which are not congruent to 1 mod q , using the same coordinate transformations as in Theorem 3.3.

The derivative $\tilde{p}'(x)$ is weakly expanding in Δ^- . To show this, let m be chosen so that $|\tilde{p}'(x) - \lambda(1 + (q+1)x^q)| < m|x|^{2q}$ on \mathbb{D}_ρ . Since $|\lambda| = 1$ and $\operatorname{Re}(x^q) > \epsilon_1|x|^q$ from Equation 23, we can estimate $|\tilde{p}'(x)|$ on Δ^- as follows:

$$\begin{aligned} |\tilde{p}'(x)| &= |1 + (q+1)x^q + \mathcal{O}(x^{2q})| \geq |1 + (q+1)x^q| - m|x|^{2q} \\ &\geq 1 + (q+1)\epsilon_1|x|^q - m|x|^{2q} > 1 + \epsilon_1|x|^q, \end{aligned}$$

for x small enough so that $|x|^q < q\epsilon_1/m$. It follows that $|\tilde{p}'(x)| > 1 + \epsilon_1|x|^q$ for $x \in \Delta^-$ and $|x|$ sufficiently small. \square

Choose $\rho' > 0$ such that the disk $\mathbb{D}_{2\rho'}(q_0)$ of radius $2\rho'$ centered at q_0 is contained in the neighborhood V_0 . We make this choice for technical reasons.

Theorem 6.2. *Let $r > 3$ be a fixed constant. There exists $\delta > 0$ such that for any $(c, a) \in \mathcal{P}_\lambda$ with $|a| < \delta$ we can find a coordinate transformation ϕ_a from a tubular neighborhood $B = \mathbb{D}_{\rho'}(q_0) \times \mathbb{D}_r$ of the local stable manifold of the semi-parabolic fixed point \mathbf{q}_a*

$$\phi_a : B \rightarrow \mathbb{D}_\rho \times \mathbb{D}_{r+\mathcal{O}(|a|)}$$

in which the Hénon map has the form $\tilde{H}_a(x, y) = (x_1, y_1)$, with

$$\begin{cases} x_1 = \lambda(x + x^{q+1} + Cx^{2q+1} + a_{2q+2}(y)x^{2q+2} + \dots) \\ y_1 = \mu y + xh(x, y) \end{cases} \quad (25)$$

and C is a constant (depending on a) and $xh(x, y) = \mathcal{O}(a)$. Moreover, the maps ϕ_a are holomorphic in a and

$$\lim_{a \rightarrow 0} \phi_a = \phi_0(x, y) = (\phi(x), y),$$

where $\phi : \mathbb{D}_{\rho'} \rightarrow \mathbb{D}_\rho$ is the change of coordinates for the polynomial $p(x) = x^2 + c_0$ with a parabolic fixed point at q_0 ,

$$\phi \circ p \circ \phi^{-1}(x) = \lambda x(1 + x^q + Cx^{2q} + \mathcal{O}(x^{2q+1})).$$

Proof. We will follow the same steps as in Section 3. The following two propositions are part of the proof.

The degenerate map $H_0(x, y) = (p(x), 0)$ has a semi-parabolic fixed point $\mathbf{q}_0 = (\frac{\lambda}{2}, 0)$ of multiplicity $q+1$ and the stable manifold $W^s(\mathbf{q}_0)$ is just a vertical line passing through \mathbf{q}_0 . The multiplicity of the semi-parabolic fixed point is constant in a neighborhood of $a = 0$ in \mathcal{P}_λ . When $a \neq 0$, $W^s(\mathbf{q}_a)$ is an analytic submanifold biholomorphic to \mathbb{C} . By [H2], $W^s(\mathbf{q}_a)$ depends analytically on a in a neighborhood of $a = 0$ inside \mathcal{P}_λ .

Definition 6.3. Denote by S_r the horizontal strip $S_r := \{(x, y) \in \mathbb{C}^2 \mid |y| < r\}$ and by $W_{loc}^s(\mathbf{q}_a)$ the connected component of $W^s(\mathbf{q}_a) \cap S_r$ that contains the fixed point \mathbf{q}_a .

Let us choose $\delta > 0$, such that for all $(c, a) \in \mathcal{P}_\lambda$ with $|a| < \delta$ the Hénon map H_a has a fixed point \mathbf{q}_a of multiplicity $q+1$ and such that the local stable manifold $W_{loc}^s(\mathbf{q}_a)$ is "vertical-like". Rigorously, we require that the horizontal distance between $W_{loc}^s(\mathbf{q}_a)$ and the vertical line that contains \mathbf{q}_a is less than $\frac{\rho'}{4}$, and that $W_{loc}^s(\mathbf{q}_a)$ has no horizontal foldings.

The parametrizing map $F_a : \mathbb{C} \rightarrow W^s(\mathbf{q}_a)$ defined in 24 is analytic in the parameter a . By Proposition 5.2, it degenerates to a translation in the horizontal direction when $a = 0$, given by $F_0(y) = \mathbf{q}_0 + (0, y)$. By Proposition 5.2 we know that

$$F_a(y) = F_0(y) + \mathcal{O}(a),$$

so F_a will map the disk $\{y \in \mathbb{C} \mid |y| < r\}$ onto a holomorphic disk inside $W^s(\mathbf{q}_a)$ around \mathbf{q}_a of size approximately $r + \mathcal{O}(a)$. For a small, fix therefore $3 < r' \leq r$ such that $W_{loc}^s(\mathbf{q}_a) \cap S_{r'} \subset F_a(S_r)$. In principle $r' = r + \mathcal{O}(|a|)$, but the vertical size is not a delicate issue, so we can think of r' as simply being r .

Proposition 6.4. *Choose $\delta > 0$ as before. For all $(c, a) \in \mathcal{P}_\lambda$ with $|a| < \delta$ there exists a coordinate transformation $\phi_a^1 : S_{r'} \rightarrow S_r$, such that in the new coordinates, the Hénon map H_a has the form $H_a(x, y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = a_1(y)x + a_2(y)x^2 + \dots \\ y_1 = \mu y + xh(x, y) \end{cases}, \quad (26)$$

where $a_j(\cdot)$ and $h(\cdot, \cdot)$ are holomorphic functions from $\{y \in \mathbb{C}, |y| < r'\}$ to \mathbb{C} , respectively from $\{(x, y) \in \mathbb{C}^2, |y| < r'\}$ to \mathbb{C} , with $a_1(0) = \lambda$ and $h(0, 0) = 0$.

Proof. Suppose $F_a(y) = (f(y), g(y))$, and let $\psi_a : S_r \rightarrow \mathbb{C}^2$ be the map

$$\psi_a(x, y) = (x + f(y), g(y)).$$

It is easy to see that ψ_a is an invertible function. The Jacobian matrix is given by

$$D\psi_a|_{(x,y)} = \begin{pmatrix} 1 & f'(y) \\ 0 & g'(y) \end{pmatrix}.$$

The local stable manifold $W_{loc}^s(\mathbf{q}_a)$ is vertical-like. In particular it has no horizontal foldings, hence $g'(y) \neq 0$ for $|y| < r$. This means that ψ_a is invertible in the strip S_r . Define $\phi_a^1(x, y) := \psi_a^{-1}(x, y)$.

The fact that $\phi_a^1(x, y)$ is holomorphic in a follows immediately, since we know that $F_a(y)$ depends holomorphically on a . From Proposition 5.2 we obtain that

$$\phi_a^1(x, y) = \phi_0^1(x, y) + \mathcal{O}(a).$$

The transformation ϕ_0^1 is straightforward to compute

$$\phi_0^1(x, y) = (x, y) - \mathbf{q}_0 = (x - \lambda/2, y).$$

In the new coordinate system the Hénon map H_0 becomes $H_0(x, y) = (x_1, y_1)$, where

$$\begin{cases} x_1 = \lambda x + x^2 \\ y_1 = 0 \end{cases}$$

Therefore, when $a \neq 0$, it is easy to control the size of the coefficients in Equation 26 in terms of a , as follows: $a_1(y) = \lambda + \mathcal{O}(a)$, $a_2(y) = 1 + \mathcal{O}(a)$, $a_i(y) = \mathcal{O}(a)$ for $i > 2$ and $h(x, y) = \mathcal{O}(a)$. \square

Proposition 6.5. *There exists $\delta > 0$ such that for all parameters $(c, a) \in \mathcal{P}_\lambda$ with $|a| < \delta$ there exists a coordinate transformation $\phi_a^2 : \mathbb{D}_{1/2} \times \mathbb{D}_r \rightarrow S_r$ in which H_a has the form $H_a(x, y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = \lambda x + a_2 x^2 + \dots + a_{2q+1} x^{2q+1} + a_{2q+2}(y) x^{2q+2} + \dots \\ y_1 = \mu y + x h(x, y) \end{cases} \quad (27)$$

where a_2 is close to 1 and the coefficients a_3, \dots, a_{2q+1} are constants close to 0.

Proof. Suppose H_a is written as in Equation 26. The proof of this proposition is the same as that of Theorem 3.2 with $m = 2q + 1$. Notice that $a_1(y) = \lambda + \mathcal{O}(a)$ for $|y| < r$, so one can perform the same change of coordinates as in the proof of Theorem 3.2

$$T_1 : (x, y) \rightarrow (u(y)x, y), \text{ where } u(y) = \prod_{n \geq 0} a_1(\mu^n y)$$

in order to set $a_1(y) = \lambda$. Since $a_1(y)$ is close to λ when $|y| < r$, it follows that the product is convergent when $|y| < r$. We get that $u(y) \neq 0$, hence $T_1(x, y)$ is invertible.

The coordinate changes that make $a_j(y)$ constant for $2 \leq j \leq 2q + 1$ are of the form

$$T_j : (x, y) \rightarrow (x + v(y)x^j, y), \text{ where } v(y) = \sum_{n \geq 0} a_j(\mu^n y) - a_j(0).$$

Clearly the sum is convergent when $|y| < r$. We get $v(y) = \mathcal{O}(a)$. The transformation T_j is invertible because x is bounded ($1/2$ would be a reasonable bound for x , but any bound less than 1 would do), so for a small $1 + v(y)jx^{j-1}$ does not vanish.

The coordinate changes that are done in order to make the first $2q + 1$ coefficients constants are identity on the second coordinate. Denote by $\phi_a^2(x, y)$ their composition. Notice also that in Equation 26, $H_0(x, y) = (\lambda x + x^2, 0)$ already has constant coefficients, so ϕ_0^2 is just the identity map. It is easy to check that

$$\phi_a^2(x, y) = (x + \mathcal{O}(a), y)$$

and $h(x, y) = \mathcal{O}(a)$. We also have that $a_2 = 1 + \mathcal{O}(a)$, $a_i = \mathcal{O}(a)$ for $2 < i \leq 2q + 1$ and $a_i(y) = \mathcal{O}(a)$ for $i > 2q + 1$. \square

We are now able to finish the proof of Theorem 6.2. The coordinate changes done in Proposition 6.4 did not require any bounds on x . The coordinate transformations done in Proposition 6.5 required only a mild assumption on x (such as $|x| < 1/2$). We will now use the coordinate changes for the polynomial p to put the Hénon map in the normal form given in Equation 25. We will thus require a bound on x comparable to the size of the normalizing neighborhood V_0 from Lemma 6.1.

Assume that H_a is written in the form 27. We use the same transformations as in Theorem 3.3 in order to eliminate the terms x^i , where $1 < i < q + 1$ and $q + 1 < i < 2q + 1$. Let $\phi_a^3 : \mathbb{D}_{2\rho' - \mathcal{O}(|a|)} \times \mathbb{D}_r \rightarrow \mathbb{D}_\rho \times \mathbb{D}_r$ denote the coordinate change from Theorem 3.3. When $a = 0$, $H_0(x, y) = (\lambda x + x^2, 0)$ and

$$\phi_0^3(x, y) = (\phi(x), y),$$

where $\phi(x)$ is the coordinate transformation used in Lemma 6.1 to put $p(x) = \lambda x + x^2$ in the normal form $\tilde{p}(x) = \lambda(x + x^{q+1} + Cx^{2q+1} + \mathcal{O}(x^{2q+2}))$.

Define $\phi_a(x, y) = \phi_a^3 \circ \phi_a^2 \circ \phi_a^1(x, y)$. Recall that ϕ_0^1 is a horizontal translation by $\lambda/2$ and ϕ_0^2 is the identity map, so when $a = 0$ the composition of the three transformations yields exactly the coordinate transformation used in Lemma 6.1 to put $p(x) = x^2 + \frac{\lambda}{2} - \frac{\lambda^2}{4}$ in the normal form $\tilde{p}(x) = \lambda(x + x^{q+1} + Cx^{2q+1} + \mathcal{O}(x^{2q+2}))$. \square **of Theorem 6.2**

In the normalizing coordinates we define attractive and repelling sectors for the Hénon map and study the behavior of \tilde{H}_a for $|a| < \delta$.

Lemma 6.6 (Attractive/Repelling sectors). *Let W^\pm be defined as in Equations 22 and 21. There exists $\rho > 0$ and $\delta > 0$ such that for all $|a| < \delta$ the derivative $D\tilde{H}_a$ expands horizontally by a factor of $(1 + \frac{\epsilon_1}{2}|x|^q)$ in the region*

$$W^- = \Delta^- \times \mathbb{D}_r = \{|x| \leq \rho \mid \operatorname{Re}(x^q) > \epsilon_0 |\operatorname{Im}(x^q)|\} \times \mathbb{D}_r$$

The compact region

$$W^+ = \Delta^+ \times \mathbb{D}_r = \{|x| \leq \rho \mid \operatorname{Re}(x^q) \leq \epsilon_0 |\operatorname{Im}(x^q)|\} \times \mathbb{D}_r$$

satisfies $\tilde{H}_a(W^+) \subset \operatorname{int}(K^+) \cup \{0\} \times \mathbb{D}_r$.

Proof. By construction, $W^+ \subset \mathcal{P}_{att} \cup \{0\} \times \mathbb{D}_r$ and all points in \mathcal{P}_{att} are attracted to the origin under forward iterations by Proposition 4.1. Hence $W^+ \subset \operatorname{int}(K^+) \cup \{0\} \times \mathbb{D}_r$. The horizontal expansion in W^- follows from Proposition 6.8 below. \square

The multiplicity of the semi-parabolic fixed point is $q + 1$, so there are exactly q connected components of W^- and q components of $\operatorname{int}(W^+)$.

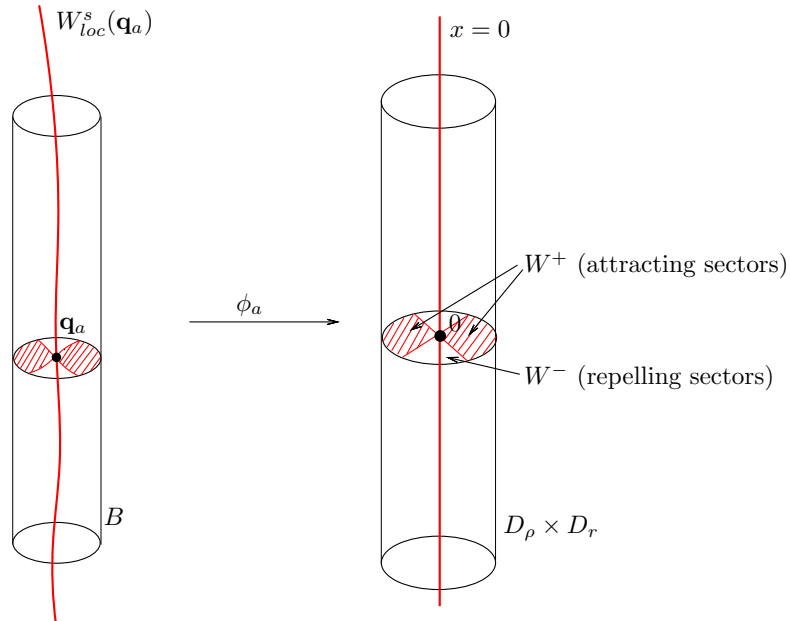


Figure 3. The transformation ϕ_a and the sectors W^\pm for $q = 2$.

Let $|a| < \delta$ as before and consider the Hénon map $\tilde{H} : \mathbb{D}_\rho \times \mathbb{D}_r \rightarrow \mathbb{C}^2$ written in normal coordinates as in Theorem 6.2,

$$\tilde{H}_a(x, y) = (\lambda(x + x^{q+1} + g_a(x, y)), \mu y + x h_a(x, y)),$$

where $g_a(x, y) = g_0(x) + \mathcal{O}(a)$ and $h_a(x, y) = \mathcal{O}(a)$ and

$$\begin{aligned} g_a(x, y) &= C_a x^{2q+1} + a_{2q+2}(y) x^{2q+2} + \dots \\ h_a(x, y) &= b_1(y) + \dots + b_k(y) x^k + \dots \end{aligned}$$

When $a = 0$, $\tilde{H}_0(x, y) = (\tilde{p}(x), 0)$, where $\tilde{p}(x) = \lambda(x + x^{q+1} + g(x))$ and

$$g(x) = C_0 x^{2q+1} + a_{2q+2} x^{2q+2} + \dots$$

The function $g_0(x, y) = g(x)$ is just a function of the variable x , hence $\partial_y g_0(x, y) \equiv 0$. For $|a| < \delta$ we can assume that there exists a constant M_a with $0 < M_a < 1$ such that

$$|\partial_y g_a(x, y)| < M_a |x|^{2q+2}. \quad (28)$$

As usual, ∂_x and ∂_y denote the partial derivatives with respect to the variable x , and respectively y . When $a = 0$ we also know that $x h_0(x, y) \equiv 0$. Moreover by the construction of the normalizing coordinates we have $x h_a(x, y) = \mathcal{O}(a)$. There exists a constant N_a , depending on a , with $0 < N_a < 1$ such that when $|a| < \delta$ the following bounds hold

$$|\partial_x(x h_a)(x, y)| < N_a \quad \text{and} \quad |\partial_y(x h_a)(x, y)| < N_a. \quad (29)$$

Let $\partial_x g_a(x, y) = x^{2q} t_a(x, y)$ and denote by m the supremum of $|t_a(x, y)|$ on the set W^- , where the supremum is taken after all $|a| < \delta$. Hence for any a with $|a| < \delta$ and any (x, y) taken from the repelling sectors $W^- = \Delta^- \times \mathbb{D}_r$ of the Hénon map we have

$$|\partial_x g_a(x, y)| < m |x|^{2q}.$$

By eventually reducing $\rho > 0$, we can assume as in Equation 23 that

$$|1 + (q+1)x^q| - m|x|^{2q} > 1 + \epsilon_1 |x|^q > 1, \quad \text{for all } x \in \Delta^-. \quad (30)$$

Definition 6.7. Let (x, y) be a point in the repelling sectors W^- of the Hénon map. Define the horizontal cone at (x, y) to be

$$\mathcal{C}_{(x,y)}^h = \{(\xi, \eta) \in T_{(x,y)} W^-, |\xi| > |\eta|\}.$$

We will show that the horizontal cones are invariant under $D\tilde{H}$ and that $D\tilde{H}$ is expanding inside the horizontal cones.

Proposition 6.8 (Horizontal cones). *Let (x, y) and (x', y') be two points from W^- such that $\tilde{H}(x, y) = (x', y')$. Then*

$$D\tilde{H}_{(x,y)} \left(\mathcal{C}_{(x,y)}^h \right) \subset \text{Int } \mathcal{C}_{(x',y')}^h$$

and $\|D\tilde{H}_{(x,y)}(\xi, \eta)\| \geq (1 + \frac{\epsilon_1}{2} |x|^q) \|(\xi, \eta)\|$ for $(\xi, \eta) \in \mathcal{C}_{(x,y)}^h$.

Proof. Pick $(\xi, \eta) \in \mathcal{C}_{(x,y)}^h$ and let $D\tilde{H}_{(x,y)}(\xi, \eta) = (\xi', \eta')$. The derivative of \tilde{H} is

$$D\tilde{H}_{(x,y)} = \begin{pmatrix} \lambda(1 + (q+1)x^q + \partial_x g_a(x, y)) & \lambda \partial_y g_a(x, y) \\ \partial_x(x h_a)(x, y) & \mu + \partial_y(x h_a)(x, y) \end{pmatrix}.$$

Consider now the Euclidean metric on the set $\mathbb{D}_\rho \times \mathbb{D}_r$ and estimate

$$|\eta'| \leq N_a |\xi| + (|\mu| + N_a) |\eta| < (2N_a + |\mu|) |\xi| \quad (31)$$

$$\begin{aligned} |\xi'| &\geq (|1 + (q+1)x^q| - m|x|^{2q}) |\xi| - M_a |x|^{2q+2} |\eta| \\ &> (|1 + (q+1)x^q| - m|x|^{2q} - M_a |x|^{2q+2}) |\xi|. \end{aligned} \quad (32)$$

We then obtain

$$|\eta'| < \frac{B_2}{B_1} |\xi'|,$$

where B_1 and B_2 are defined in the obvious way

$$\begin{aligned} B_2 &:= 2N_a + |\mu| \\ B_1 &:= |1 + (q+1)x^q| - m|x|^{2q} - M_a |x|^{2q+2}. \end{aligned}$$

The bounds N_a , M_a and $|\mu| = |a|^2$ tend to 0 as $a \rightarrow 0$, so we can make B_2 as small as we want, for example we assume $B_2 < \frac{1}{2}$. The points (x, y) and (x', y') are chosen from the repelling sectors, so we can assume that

$$B_1 \geq 1 + \frac{\epsilon_1}{2} |x|^q. \quad (33)$$

In conclusion we get $|\eta'| < B_2 |\xi'|$, so $(\xi', \eta') \in \text{Int } \mathcal{C}_{(x', y')}^h$. In fact, when $\eta = 0$, we have that $|\eta'| < N_a |\xi'|$, which will be useful in Lemma 10.5.

The same computation 32 also shows that $|\xi'| > B_1 |\xi|$ so $D\tilde{H}$ expands the horizontal length of vectors, i.e.

$$\|(\xi', \eta')\| = \max\{|\xi'|, |\eta'|\} = |\xi'| > B_1 |\xi| \geq |\xi| = \max\{|\xi|, |\eta|\} = \|(\xi, \eta)\|. \quad (34)$$

□

7. CONSTRUCTION OF A NEIGHBORHOOD V FOR J^+

We will build a neighborhood of J^+ for a semi-parabolic Hénon map H_a inside a polydisk $\mathbb{D}_r \times \mathbb{D}_r$, inspired by the construction of a neighborhood of the Julia set of a parabolic polynomial p on which p is strictly, but not strongly expanding, as in [DH]. Inside a tubular neighborhood B of the local stable manifold $W^s(\mathbf{q}_a)$, we want to forget about the dynamics of the polynomial p and construct a neighborhood of $J^+ \cap B$ that is meaningful for the dynamics of the Hénon map.

Let q_0 be the parabolic fixed point of the polynomial p and let q_1 be the other preimage of q_0 under p . Suppose $|a| < \delta$ and consider $B = \mathbb{D}_{\rho'}(q_0) \times \mathbb{D}_r$ as defined in Theorem 6.2. Let W^\pm be the attractive/repelling sectors from Lemma 6.6. Define $W_B^\pm = \phi_a^{-1}(W^\pm) \cap \mathbb{D}_r \times \mathbb{D}_r$. For our purpose, it is more convenient to view these sectors in B rather than in the normalized coordinates. The set $H^{-1}(B) \cap \mathbb{D}_r \times \mathbb{D}_r$ consists of two connected components. We denote by B' the connected component which contains q_1 . Define $W_{B'}^\pm$ to be the preimage of the attractive/repelling sectors W_B^\pm in B' .

Choose $\rho'' > 0$ as large as possible so that $p^{\circ 2}(\mathbb{D}_{\rho''}(q_0)) \subset \mathbb{D}_{\rho'}(q_0)$. Clearly this choice depends only on the parabolic polynomial p and the radius ρ' . Consider the annulus $A := A(q_0; \rho', \rho'')$ between the disk of radius ρ'' and the disk of radius ρ' centered at q_0 . Let n be the first iterate of p such that $p^{\circ(n+1)}(0) \in \mathbb{D}_{\rho'}(q_0)$ and implicitly $p^{\circ(n+1)}(0) \in A$.

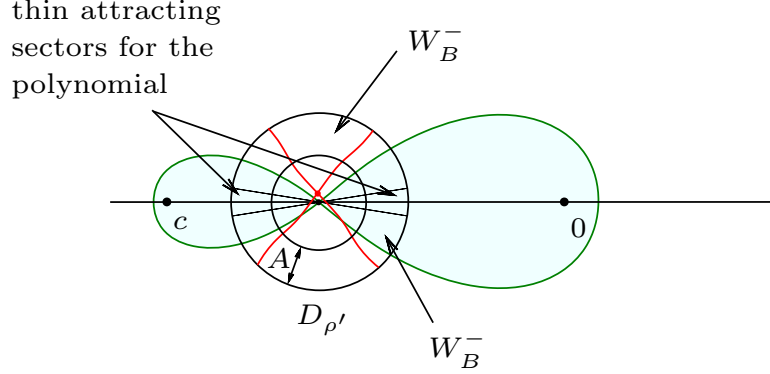


Figure 4. Here $q = 2$. This is a cross section around the parabolic fixed point of the polynomial $p(x) = x^2 + c_0$. The red lines are the boundaries of the attractive sectors for the Hénon map. The thin attractive sectors for the polynomial and their preimages are shown in green.

We now construct attractive sectors S_{att} associated with the parabolic polynomial p in $\mathbb{D}_{\rho'}(q_0)$, thin enough along the attractive axes of the polynomial so that

$$(p^{-\circ(n+1)}(S_{att}) \cap A) \times \mathbb{D}_r \subset W_B^+. \quad (35)$$

Let $\partial^{in}(W_B^+)$ be the part of the boundary of W_B^+ that lies strictly inside $A \times \mathbb{D}_r$. Similarly, let $\partial^{in}(p^{-\circ(n+1)}(S_{att}) \cap A)$ be the part of the boundary of $p^{-\circ(n+1)}(S_{att}) \cap A$ that is strictly inside the annulus A . We will further require that S_{att} be thin enough so that the distance between the two boundaries $\partial^{in}(W_B^+)$ and $\partial^{in}(p^{-\circ(n+1)}(S_{att}) \cap A) \times \mathbb{D}_r$ is at least $\eta_0 > 0$. The constant η_0 depends only on the local dynamics of the polynomial p and it can be taken to be a fraction of the distance between $\partial^{in}(\Delta^+ \cap A)$ and an attractive axes that passes through Δ^+ , where Δ^+ is defined in Lemma 6.1.

Let $\Omega = p^{-\circ n}(S_{att})$. In the definition of the set Ω , we only consider the preimages of S_{att} that contain the parabolic fixed point in the boundary, so they are local preimages. The set Ω has q connected components and contains the critical value, but not the critical point of the polynomial p . Let us now define a set U as the complement of Ω inside an equipotential of the Green's function of p , i.e.

$$U := \mathbb{C} - \Omega - \{z \in \mathbb{C} - K_p \mid |\Phi_p^{-1}(z)| \geq R\} \quad (36)$$

for some large enough $R > 2$. Define $U' := p^{-1}(U)$ and $\Omega' := p^{-1}(\Omega)$. The constant R is chosen so that the outer boundary of U' (which is an equipotential of the polynomial p) is in the escaping set U^+ (more precisely in the set V^+).

We endow U' with the Poincaré metric of U . The set U' is contained in U and $p : U' \rightarrow U$ is a covering map, hence expanding. However U' is not relatively compact in U , so there is no constant of uniform expansion.

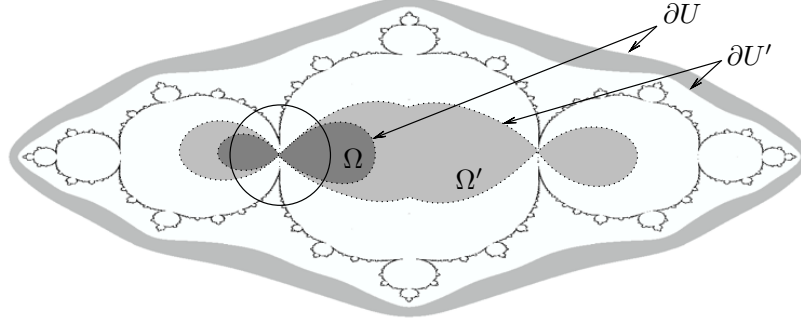


Figure 5. The polynomial $p(x) = x^2 - \frac{3}{4}$ has a parabolic fixed point at $-\frac{1}{2}$ and locally connected Julia set J_p . The corresponding neighborhoods U and U' are also shown, but U' is not compactly contained in U , as in the hyperbolic case. Their boundaries touch at the parabolic fixed point.

Define

$$V := (U' \times \mathbb{D}_r - (B \cup B')) \cup (W_B^- \cup W_{B'}^-), \quad (37)$$

where B and B' are the two tubular neighborhoods defined above.

The vertical size of the neighborhood V is $r > 3$ where r is chosen so that $\mathbb{D}_r \subset U' \cup \Omega'$. We also require that $\overline{H(V)}$ does not intersect the horizontal boundary of V , that is $|ax| < r$ for any $x \in U'$.

The horizontal size of the neighborhood V is given by an equipotential of the parabolic polynomial, contained entirely in the escaping set U^+ .

Let a be small enough so that the following two conditions hold:

- $r|a| < |p(x) - c_0|$ for any x in U' . This is possible because we removed a disc around the critical value c_0 of the polynomial p , hence $\inf_{x \in U'} |p(x) - c_0| > 0$.
- $2r|a| < d(\partial U' - \mathbb{D}_{\rho'}(q_0), \partial U)$. This assures that for all x in $U' - \mathbb{D}_{\rho'}(q_0)$ the disk of radius $2r|a|$ around x belongs to U . In other words, the $2r|a|$ -neighborhood of the set $U' - \mathbb{D}_{\rho'}(q_0)$ is compactly contained in U .

Furthermore, we choose a small enough so that in the construction of the set V we make sure to remove points only from the interior of K^+ and not J^+ . We only need to check for points that are outside the tubes B and B' . This is guaranteed by the following lemma.

Lemma 7.1. *The removed set $\Omega' \times \mathbb{D}_r - (B \cup B')$ belongs to the interior of K^+ .*

Proof. From the construction of the sets U' and Ω' , after at most $n + 1$ iterations of the polynomial p , points from Ω' are mapped to $\mathbb{D}_{\rho'}(q_0)$. In fact, points from $\Omega - \mathbb{D}_{\rho'}(q_0)$ are mapped to the region $p^{-o(n+1)}(S_{att}) \cap A$ inside the annulus A after at most $n + 1$ iterates.

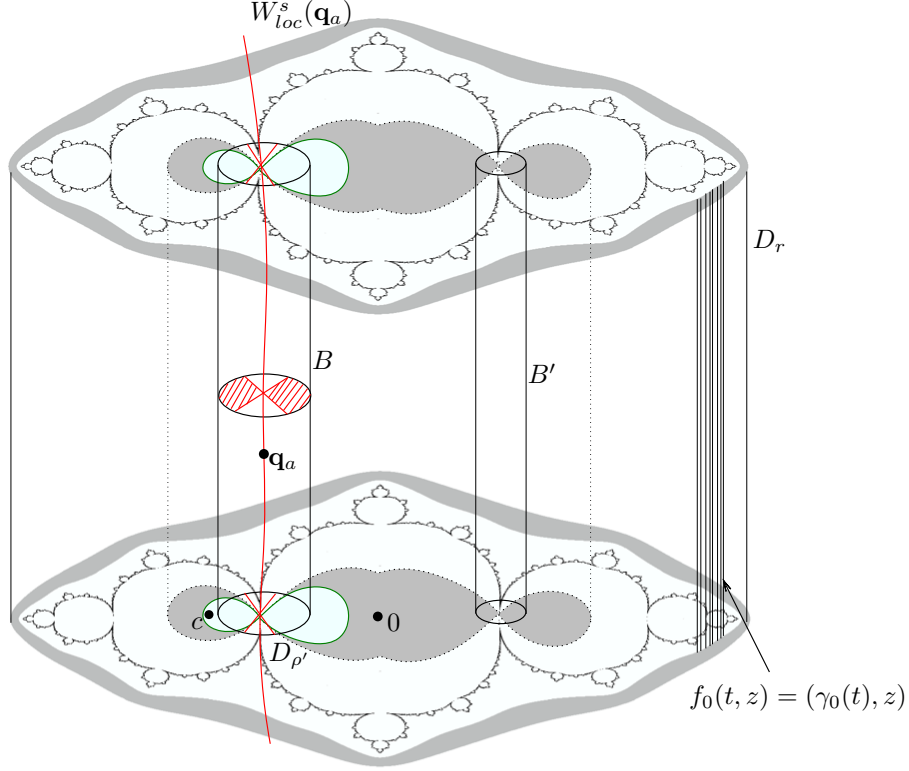


Figure 6. A neighborhood V of the set J^+ in $\mathbb{D}_r \times \mathbb{D}_r$. The map γ_0 used in the definition of the fiber $f_0(t, z)$ is the equipotential that gives the outer boundary of the set U' (see also Equation 60).

The Hénon map is given by $H(x, y) = (p(x) + a^2w + ay, ax)$. The y component does not pose any problems as $|ax| < r$ for any $x \in \Omega'$. Let $(x, y) \in (\Omega - \mathbb{D}_{\rho'}(q_0)) \times \mathbb{D}_r$. Suppose k is the first iterate for which $p^{\circ k}(x) \in \mathbb{D}_{\rho'}(q_0)$. Then $p^{\circ k}(x) \in p^{-\circ(n+1-k)}(S_{att}) \cap A$. After k iterates, the distance between the x coordinate of $H^{\circ k}(x, y)$ and $p^k(x)$ is at most $|a|\eta_1$, where η_1 is a constant which depends only on the parabolic polynomial p and the integer n . Notice that n is fixed and depends only on the polynomial p and ρ' , which is also fixed. However, if after $n+1$ iterates $p^{\circ(n+1)}(x)$ is too close to the outer boundary of A , then we take one more iterate and it is still in A , by construction of this annulus. Based on the construction of S_{att} we know that for $|a| < \eta_0/\eta_1$, $H^{\circ k}(x, y) \in W_B^+ \cap A \times \mathbb{D}_r$, which is in the interior of K^+ .

This shows that $\Omega \times \mathbb{D}_r - B$ belongs to the interior of K^+ , but a similar technical argument can be made for $\Omega' \times \mathbb{D}_r - (B \cup B')$. \square

Let \overline{V} denote the set V together with $W_{loc}^s(q_a)$ and $H^{-1}(W_{loc}^s(q_a)) \cap B'$. In all other cases \overline{X} denotes the closure of the set X .

Lemma 7.2. $J^+ \cap (\mathbb{D}_r \times \mathbb{D}_r) = J^+ \cap \bar{V}$.

Proof. The outer boundary of the set V is an equipotential of the polynomial cross \mathbb{D}_r , which belongs to U^+ . From the tubular neighborhood B of the local stable manifold we removed only the attractive sectors W_B^+ , which are contained inside the interior of K^+ union the local stable manifold $W_{loc}^s(\mathbf{q}_a)$. From B' we only removed the attractive sectors $W_{B'}^+$, which are contained inside the interior of K^+ union a preimage of the local stable manifold $H^{-1}(W_{loc}^s(\mathbf{q}_a)) \cap B'$. Outside of $B \cup B'$, we removed the set $\Omega' \times \mathbb{D}_r - (B \cup B')$ which belongs to the interior of K^+ , as shown in Lemma 7.1. Therefore

$$J^+ \cap (\mathbb{D}_r \times \mathbb{D}_r) = (J^+ \cap V) \cup W_{loc}^s(\mathbf{q}_a) \cup (H^{-1}(W_{loc}^s(\mathbf{q}_a)) \cap B') = J^+ \cap \bar{V}.$$

In this sense we say that \bar{V} is a neighborhood of J^+ inside the bidisk $\mathbb{D}_r \times \mathbb{D}_r$. \square

Corollary 7.2.1. $H(J^+ \cap \bar{V}) \subset J^+ \cap \bar{V}$ and $H^{-1}(J^+ \cap \bar{V}) \cap (\mathbb{D}_r \times \mathbb{D}_r) \subset J^+ \cap \bar{V}$.

Lemma 7.3. $J^+ \cap \bar{V} = \bigcap_{n \geq 0} H^{-on}(\bar{V} \cap \bar{U}^+)$.

Proof. Let $q \in \bigcap_{n \geq 0} H^{-on}(\bar{V} \cap \bar{U}^+)$, where $\bar{U}^+ = U^+ \cup J^+$. Since all forward iterates of q remain in the bounded set \bar{V} , q cannot belong to U^+ . Hence $q \in J^+$. Suppose now that $q \in J^+ \cap \bar{V}$. By construction of the neighborhood V , $H(J^+ \cap \bar{V}) \subset J^+ \cap \bar{V}$, so all forward iterates of q remain in \bar{V} . Hence $q \in \bigcap_{n \geq 0} H^{-on}(\bar{V})$. \square

From Proposition 7.2 we immediately get that the Julia set $J = \bigcap_{n \geq 0} H^{on}(J^+ \cap \bar{V})$.

8. INFINITESIMAL METRICS ON V

On the set V we will define two infinitesimal metrics with respect to which the derivative of the Hénon map is weakly expanding horizontally and strongly contracting vertically.

To formalize our definitions, recall that q_0 is the parabolic fixed point of the quadratic polynomial p , $B = \mathbb{D}_{\rho'}(q_0) \times \mathbb{D}_r$ and $B'' = \mathbb{D}_{\rho''}(q_0) \times \mathbb{D}_r$, where $0 < \rho'' < \rho'$. We have chosen a small enough so that the local stable manifold $W_{loc}^s(\mathbf{q}_a)$ of the semi-parabolic fixed point \mathbf{q}_a is contained in B'' and that Equation 35 is satisfied. In addition, the set U' is compactly contained in U outside the disk $\mathbb{D}_{\rho''}(q_0)$.

Definition 8.1 (Euclidean metric). In the repelling sectors W_B^- of the tubular neighborhood B of the local stable manifold of the semi-parabolic fixed point, we have a natural metric defined as a pull-back of the Euclidean metric from the normalizing coordinates by $\phi_a : W_B^- \rightarrow W^- \subset \mathbb{D}_\rho \times \mathbb{D}_r$,

$$\mu_B((x, y), (\xi, \eta)) := \max \left\{ |\tilde{\xi}|, |\tilde{\eta}| \right\}, \quad (38)$$

where ϕ_a is the coordinate transformation from Lemma 6.2, $(\tilde{\xi}, \tilde{\eta}) = D\phi_a|_{(x,y)}(\xi, \eta)$ and $|\tilde{\xi}|$ and $|\tilde{\eta}|$ represent the length of $\tilde{\xi}$ and $\tilde{\eta}$ with respect to the Euclidean metric.

Remark 8.2. By construction, the coordinate transformation ϕ_a takes horizontal curves to horizontal curves. Therefore, if we choose a point $(x, y) \in B$ and a tangent vector $(\xi, 0) \in T_{(x,y)}B$, then $D\phi_a|_{(x,y)}(\xi, 0) = (\tilde{\xi}, 0)$ and $\mu_B((x, y), (\xi, 0)) = |\tilde{\xi}|$.

Definition 8.3 (Poincaré metric). The set V , outside of a small neighborhood B'' of the local stable manifold $W_{loc}^s(\mathbf{q}_a)$ of the semi-parabolic fixed point \mathbf{q}_a , is contained in the product space $U \times \mathbb{D}_r$. On $V - B''$ we will use a product metric $\mu_U \times \mu_E$ of the Poincaré metric μ_U of the set U and the Euclidean metric μ_E on the vertical disk \mathbb{D}_r . Tangent vectors (ξ, η) from $T_{(x,y)}V - B''$ will be measured with respect to the metric

$$\mu_P((x, y), (\xi, \eta)) := \max(\mu_U(x, \xi), |\eta|), \quad (39)$$

where $|\eta|$ is the absolute value of the complex number η .

Definition 8.4 (Combining the metrics). Let $B' = (H^{-1}(B) - B) \cap V$ be one of the preimages of B in V as in Section 7. Choose a number M such that

$$M \geq \sup_{\substack{(x,y) \in B' \\ (\xi,\eta) \in \mathcal{C}_{(x,y)}^{h,P}}} \frac{2 \cdot \mu_P((x, y), (\xi, \eta))}{\mu_B(H(x, y), DH_{(x,y)}(\xi, \eta))}. \quad (40)$$

Define as in [DH] $\mu := \inf\{\mu_P, M\mu_B\}$, where the infimum is taken pointwise between the metrics on V .

Remark 8.5. Note that the supremum from 40 is a finite number. Since $(\xi, \eta) \in \mathcal{C}_{(x,y)}^{h,P}$, the numerator $\mu_P((x, y), (\xi, \eta))$ is equal to $\mu_U(x, \xi)$ which is bounded above for all (x, y) in B' , because B' is far away from the boundary of the set U , so the Poincaré metric μ_U is finite on B' . The denominator is bounded away from zero, because the tangent vector $DH_{(x,y)}(\xi, \eta)$ belongs to the horizontal cone $\mathcal{C}^{h,B}(H(x, y))$ when the vector (ξ, η) belongs to the horizontal cone $\mathcal{C}_{(x,y)}^{h,P}$.

Remark 8.6. The constant M is chosen so that the Hénon map expands in horizontal cones with respect to the combined metrics, as we will see in Theorem 8.7. By eventually increasing M we can assume that inside horizontal cones on $\partial\mathbb{D}_{\rho'}(q_0) \times \mathbb{D}_r$, the infimum of the two metrics is attained by the Poincaré metric μ_P . As we approach the point q_0 which belongs to the boundary of U , the Poincaré metric explodes in horizontal cones, whereas the pull-back Euclidean metric μ_B is finite, so the infimum of the two metrics will be realized by the metric μ_B . By eventually reducing a , we can assume that inside horizontal cones on $\partial\mathbb{D}_{\rho'}(\mathbf{q}_0) \times \mathbb{D}_r$, the infimum is attained by the pull-back metric μ_B .

Theorem 8.7 (μ -Expansion). Consider $(x, y), (x_1, y_1) \in V$ with $H(x, y) = (x_1, y_1)$. Let (ξ, η) and (ξ_1, η_1) be two tangent vectors such that $DH_{(x,y)}(\xi, \eta) = (\xi_1, \eta_1)$ and (ξ, η) belongs to the horizontal cones defined at (x, y) , i.e. $\mathcal{C}_{(x,y)}^{h,P}$ and/or $\mathcal{C}_{(x,y)}^{h,B}$.

Then the Hénon map is strictly but not strongly expanding with respect to μ , that is

$$\mu((x_1, y_1), (\xi_1, \eta_1)) > \alpha(x, y) \cdot \mu((x, y), (\xi, \eta)), \quad \text{where } \alpha(x, y) > 1,$$

and $\alpha(x, y)$ is a constant in cases (a), (c) and (d), and $\alpha(x, y) = \mathcal{E}(x, y)$, the expansion factor of the pull-back metric μ_B , in case (b). The expansion factor $\alpha(x, y) \rightarrow 1$ if and only if (x, y) tends to $W_{loc}^s(\mathbf{q}_a)$, the local stable manifold of the semi-parabolic fixed point.

Proof. There are four cases to consider:

(a) Suppose

$$\begin{aligned}\mu((x, y), (\xi, \eta)) &= \mu_P((x, y), (\xi, \eta)) \quad \text{and} \\ \mu((x_1, y_1), (\xi_1, \eta_1)) &= \mu_P((x_1, y_1), (\xi_1, \eta_1)).\end{aligned}$$

Since the Poincaré metric is smaller than the pull-back metric, it means that the points (x, y) and (x_1, y_1) are not very close to $q_0 \times \mathbb{D}_r$. In particular by Remark 8.6 they must lie outside $\mathbb{D}_{\rho''}(q_0) \times \mathbb{D}_r$. By Proposition 9.8,

$$\mu((x_1, y_1), (\xi_1, \eta_1)) > k \cdot \mu((x, y), (\xi, \eta)).$$

(b) Suppose

$$\begin{aligned}\mu((x, y), (\xi, \eta)) &= M\mu_B((x, y), (\xi, \eta)) \quad \text{and} \\ \mu((x_1, y_1), (\xi_1, \eta_1)) &= M\mu_B((x_1, y_1), (\xi_1, \eta_1)).\end{aligned}$$

By Proposition 6.6 we get $\mu((x_1, y_1), (\xi_1, \eta_1)) > \mathcal{E}(x, y) \cdot \mu((x, y), (\xi, \eta))$. The expansion factor $\mathcal{E}(x, y) = 1 + \frac{\epsilon_1}{2}|\tilde{x}|^q$, where $(\tilde{x}, \tilde{y}) = \phi_a(x, y)$.

(c) Suppose

$$\begin{aligned}\mu((x, y), (\xi, \eta)) &= M\mu_B((x, y), (\xi, \eta)) \quad \text{and} \\ \mu((x_1, y_1), (\xi_1, \eta_1)) &= \mu_P((x_1, y_1), (\xi_1, \eta_1)).\end{aligned}$$

By Remark 8.6 above, the point (x_1, y_1) cannot be too close to $q_0 \times \mathbb{D}_r$ and it must stay outside the small tube B'' . By Proposition 9.8, we have

$$\begin{aligned}\mu_P((x_1, y_1), (\xi_1, \eta_1)) &> k \cdot \mu_P((x, y), (\xi, \eta)) \\ &\geq k \cdot M\mu_B((x, y), (\xi, \eta)) = k \cdot \mu((x, y), (\xi, \eta)).\end{aligned}$$

(d) Suppose

$$\begin{aligned}\mu((x, y), (\xi, \eta)) &= \mu_P((x, y), (\xi, \eta)) \quad \text{and} \\ \mu((x_1, y_1), (\xi_1, \eta_1)) &= M\mu_B((x_1, y_1), (\xi_1, \eta_1)).\end{aligned}$$

In this case there are two subcases to consider:

(i) If $(x, y) \in B'$, then by the choice of the constant M we have

$$\begin{aligned}\mu_P((x, y), (\xi, \eta)) &< \frac{2 \cdot \mu_P((x, y), (\xi, \eta))}{\mu_B((x_1, y_1), (\xi_1, \eta_1))} \cdot \frac{1}{2} \mu_B((x_1, y_1), (\xi_1, \eta_1)) \\ &< \frac{1}{2} \cdot M\mu_B((x_1, y_1), (\xi_1, \eta_1))\end{aligned}$$

hence $\mu((x_1, y_1), (\xi_1, \eta_1)) > 2 \cdot \mu((x, y), (\xi, \eta))$.

(ii) If $(x, y) \in B$, and the Poincaré metric is smaller than the pull-back metric, then (x, y) must be outside the small tube B'' which encloses the local stable manifold $W_{loc}^s(\mathbf{q}_a)$. If we denote by $k' := \inf_{\substack{(x, y) \in V-B'' \\ |a| < \delta}} \mathcal{E}(x, y)$ the infimum of

the expansion rate $\mathcal{E}(x, y)$ outside B'' , then $k' > 1$. By Proposition 6.6, we know that $\mu_B((x_1, y_1), (\xi_1, \eta_1)) > \mathcal{E}(x, y) \cdot \mu_B((x, y), (\xi, \eta))$. Therefore

$$\begin{aligned}M\mu_B((x_1, y_1), (\xi_1, \eta_1)) &> \mathcal{E}(x, y) \cdot M\mu_B((x, y), (\xi, \eta)) \\ &\geq k' \cdot \mu_P((x, y), (\xi, \eta)),\end{aligned}$$

hence $\mu((x_1, y_1), (\xi_1, \eta_1)) > k' \cdot \mu((x, y), (\xi, \eta))$. \square

9. VERTICAL AND HORIZONTAL CONES

Definition 9.1. In Section 6 we gave the definition 6.7 of a horizontal cone at a point (x, y) from the set $\mathbb{D}_\rho \times \mathbb{D}_r$, namely:

$$\mathcal{C}_{(x,y)}^h = \{(\xi, \eta) \in T_{(x,y)}\mathbb{D}_\rho \times \mathbb{D}_r, |\xi| > |\eta|\}.$$

We will now define the vertical cone at a point (x, y) from the set $\mathbb{D}_\rho \times \mathbb{D}_r$ to be

$$\mathcal{C}_{(x,y)}^v = \{(\xi, \eta) \in T_{(x,y)}\mathbb{D}_\rho \times \mathbb{D}_r, |\xi| < |x|^{2q}|\eta|\}.$$

In Proposition 6.8, we showed that horizontal cones are invariant under $D\tilde{H}$. Moreover, by Equation 34 of Proposition 6.6, the Hénon map expands the length of vectors from the horizontal cone $\mathcal{C}_{(x,y)}^h$ by a factor of $1 + \frac{\epsilon_1}{2}|x|^q$. Now we will show that the vertical cones are invariant under $D\tilde{H}^{-1}$ and that $D\tilde{H}^{-1}$ is expanding in the vertical direction.

Proposition 9.2 (Vertical cones). *Consider (x, y) and (x_1, y_1) in the repelling sectors of $\mathbb{D}_\rho \times \mathbb{D}_r$ such that $\tilde{H}(x, y) = (x_1, y_1)$. Then*

$$D\tilde{H}_{(x_1,y_1)}^{-1}(\mathcal{C}_{(x_1,y_1)}^v) \subset \text{Int } \mathcal{C}_{(x,y)}^v$$

and $\|D\tilde{H}_{(x_1,y_1)}^{-1}(\xi', \eta')\| \geq \frac{1}{|a|^2+1/2}\|(\xi', \eta')\|$ for $(\xi', \eta') \in \mathcal{C}_{(x_1,y_1)}^v$.

Proof. Let $(\xi', \eta') \in \mathcal{C}_{(x_1,y_1)}^v$ and $(\xi, \eta) = D\tilde{H}_{(x_1,y_1)}^{-1}(\xi', \eta')$. We need to show that $(\xi, \eta) \in \mathcal{C}_{(x,y)}^v$ so we compute as before

$$\begin{aligned} \xi' &= \lambda(1 + (q+1)x^q + \partial_x g_a(x, y))\xi + \lambda\partial_y g_a(x, y)\eta \\ \eta' &= \partial_x(xh_a)(x, y)\xi + (\mu + \partial_y(xh_a)(x, y))\eta \end{aligned}$$

and estimate

$$\begin{aligned} |\xi'| &> (|1 + (q+1)x^q| - m|x|^{2q})|\xi| - M_a|x|^{2q+2}|\eta| \\ |\eta'| &< N_a|\xi| + (|\mu| + N_a)|\eta|. \end{aligned}$$

Since (ξ', η') belongs to the vertical cone at (x_1, y_1) , we also know that

$$|\xi'| < |x_1|^{2q}|\eta'| < |x|^{2q}|1 + x^q + g_a(x, y)/x|^{2q}|\eta'| < |x|^{2q}M_1^{2q}|\eta'|,$$

where M_1 is the supremum of $|1 + x^q + g_a(x, y)/x|$ on the repelling sectors W^- of the tubular neighborhood B , that is

$$M_1 := \sup_{(x,y) \in W^-, |a| < \delta} |1 + x^q + g_a(x, y)/x|. \quad (41)$$

Clearly $M_1 > 0$. In fact we could take a constant $M_1 > 1$ because $Re(x^q) > \epsilon_1|x|^q$ in the repelling sectors W^- . By combining these inequalities we get

$$(|1 + (q+1)x^q| - m|x|^{2q})|\xi| - M_a|x|^{2q+2}|\eta| < M_1^{2q}N_a|x|^{2q}|\xi| + M_1^{2q}(|\mu| + N_a)|x|^{2q}|\eta|.$$

After regrouping the terms, we obtain

$$|\xi| < \frac{A_2}{A_1} |x|^{2q} |\eta|$$

where A_1 and A_2 are defined as follows

$$\begin{aligned} A_1 &:= |1 + (q+1)x^q| - (m + M_1^{2q} N_a) |x|^{2q} \\ A_2 &:= M_1^{2q} (|\mu| + N_a) + M_a |x|^2. \end{aligned}$$

Since x is chosen from the repelling sectors we have $|1 + (q+1)x^q| - m|x|^{2q} > 1 + \epsilon_1 |x|^q$. The bounds N_a , M_a and the eigenvalue μ all depend on a , and they tend to 0 as $a \rightarrow 0$, so for $|a|$ small we can assume that $A_1 > \frac{2}{3}$ and $A_2 < \frac{1}{3}$. Hence $(\xi, \eta) \in \text{Int } \mathcal{C}_{(x,y)}^v$.

We will now show that inside the vertical cones, $D\tilde{H}^{-1}$ is expanding with respect to the Euclidean metric. We have

$$\begin{aligned} |\eta'| &< N_a |\xi| + (|\mu| + N_a) |\eta| < N_a \frac{A_2}{A_1} |x|^{2q} |\eta| + (|\mu| + N_a) |\eta| \\ &< \left(\frac{1}{2} N_a |x|^{2q} + |\mu| + N_a \right) |\eta| < \left(|\mu| + \frac{3}{2} N_a \right) |\eta|, \end{aligned}$$

since $|x| < \rho < 1$. We obtain $|\eta| > \frac{1}{|\mu| + \frac{3}{2} N_a} |\eta'|$. For a sufficiently small we can assume that $N_a < 1/3$, therefore $|\eta| > \frac{1}{|\mu| + 1/2} |\eta'| = \frac{1}{|a|^2 + 1/2} |\eta'|$.

Since (ξ, η) and (ξ', η') belong to vertical cones, we have $\|(\xi, \eta)\| = \max(|\xi|, |\eta|) = |\eta|$ and $\|(\xi', \eta')\| = \max(|\xi'|, |\eta'|) = |\eta'|$, hence $D\tilde{H}^{-1}$ expands in the vertical cones with a factor strictly greater than 1. \square

The vertical cones that we have introduced in Definition 9.1, are taken with respect to the Euclidean metric, in the normalized coordinates around the local stable manifold of the semi-parabolic fixed point.

Definition 9.3. Let $\mathcal{C}_{(x,y)}^{v,B} := D\phi_a^{-1} \left(\mathcal{C}_{\phi_a(x,y)}^v \right)$ and $\mathcal{C}_{(x,y)}^{h,B} := D\phi_a^{-1} \left(\mathcal{C}_{\phi_a(x,y)}^h \right)$ denote the pull-back of the vertical cone $\mathcal{C}_{\phi_a(x,y)}^v$ and respectively of the horizontal cone $\mathcal{C}_{\phi_a(x,y)}^h$ defined in 9.1, from the normalized coordinates $\mathbb{D}_\rho \times \mathbb{D}_r$ into B , by the change of coordinate function ϕ_a .

On the set V , outside of a small neighborhood $\mathbb{D}_{\rho''}(q_0) \times \mathbb{D}_r$ of the local stable manifold $W_{loc}^s(\mathbf{q}_a)$ of the semi-parabolic fixed point \mathbf{q}_a , we will use a product metric $\mu_U \times \mu_E$ of the Poincaré metric μ_U of the set U and the Euclidean metric μ_E on the vertical disk \mathbb{D}_r . We will also define vertical and horizontal cones with respect to the product metric and show invariance under DH^{-1} , respectively under DH .

Let us notice first that $p : U' \rightarrow U$ is a covering map, hence a local isometry from the set U' endowed with the Poincaré metric of U' to the set U endowed with the Poincaré metric of U . Since U' is contained in U , the inclusion map is contracting with respect to the Poincaré metrics of U' and U . Therefore, if we consider $x \in U'$ and ξ a complex tangent vector, then

$$\mu_U(p(x), p'(x)\xi) = \mu_{U'}(x, \xi) > \mu_U(x, \xi), \quad (42)$$

hence the polynomial p is expanding with respect to the Poincaré metric of U . The boundaries of the sets U and U' touch at the parabolic fixed point q_0 , so there is no constant of uniform expansion. However, the set $U' - \mathbb{D}_{\rho''}(q_0)$ is compactly contained in U , therefore there exists a constant $k_0 > 1$ such that

$$\mu_U(p(x), p'(x)\xi) > k_0 \cdot \mu_U(x, \xi), \quad \text{for all } x \in U' - \mathbb{D}_{\rho''}(q_0). \quad (43)$$

The constant k_0 from inequality 43 is just $\liminf_{x \in U' - \mathbb{D}_{\rho''}(q_0)} \frac{\rho_U(p(x))|p'(x)|}{\rho_U(x)}$, where ρ_U is the density function of the Poincaré metric μ_U , that is ρ_U is a positive continuous function such that

$$\mu_U(x, \xi) = \rho_U(x)|\xi|, \quad \text{for all } x \in U \text{ and } \xi \in T_x U.$$

By inequality 42, we know that $k_0 \geq 1$. Suppose by contradiction that $k_0 = 1$. There exists a sequence of points $\{x_n\}_{n \geq 1}$ in $U' - \mathbb{D}_{\rho''}(q_0)$ such that $\frac{\rho_U(p(x_n))|p'(x_n)|}{\rho_U(x_n)} \rightarrow 1$. The sequence x_n is bounded, therefore there exists a convergent subsequence $x_{n_k} \rightarrow x^*$, where x^* belongs to the closure of $U' - \mathbb{D}_{\rho''}(q_0)$. However, the limit point x^* cannot belong to the part of the boundary of $U' - \mathbb{D}_{\rho''}(q_0)$ given by $\partial U' - \mathbb{D}_{\rho''}(q_0)$, because the latter is compactly contained in U , so $\rho_U(x^*)$ is finite. However $p(\partial U' - \mathbb{D}_{\rho''}(q_0)) \subset \partial U$, so the Poincaré metric of $\rho_U(p(x^*))$ would be infinite. Note also that $|p'(x^*)|$ is different from 0, as the closure of U' does not contain the critical point 0 of the polynomial p . This contradicts the fact that $\lim_{k \rightarrow \infty} \frac{\rho_U(p(x_{n_k}))|p'(x_{n_k})|}{\rho_U(x_{n_k})} = 1$. The only possibility left is that x^* belongs to U' , but then, by inequality 42 we know that $\rho_U(p(x^*))|p'(x^*)| > \rho_U(x^*)$. In conclusion k_0 must also be strictly greater than 1.

Lastly, we also make the observation that on the set $U' - \mathbb{D}_{\rho''}(q_0)$, the Poicaré metric μ_U is bounded above and below by the Euclidean metric, that is, there exist two positive constants m_1 and m_2 such that $m_1 < \rho_U(x) < m_2$ for any $x \in U' - \mathbb{D}_{\rho''}(q_0)$.

Definition 9.4. Let $\tau < 1$ to be chosen later. Define the vertical cone at a point (x, y) from the set $U' \times \mathbb{D}_r - \mathbb{D}_{\rho''}(q_0) \times \mathbb{D}_r$ to be

$$\mathcal{C}_{(x,y)}^{v,P} = \{(\xi, \eta) \in T_{(x,y)}U' \times \mathbb{D}_r, \quad \mu_U(x, \xi) < \tau|\eta|\}.$$

Define the horizontal cone at a point (x, y) from the set $U' \times \mathbb{D}_r - \mathbb{D}_{\rho''}(q_0) \times \mathbb{D}_r$ as

$$\mathcal{C}_{(x,y)}^{h,P} = \{(\xi, \eta) \in T_{(x,y)}U' \times \mathbb{D}_r, \quad \mu_U(x, \xi) > |\eta|\}.$$

Proposition 9.5 (Vertical cones). Consider (x, y) and (x', y') in $U' \times \mathbb{D}_r - \mathbb{D}_{\rho''}(q_0) \times \mathbb{D}_r$ such that $H(x', y') = (x, y)$. Then

$$DH_{(x,y)}^{-1}(\mathcal{C}_{(x,y)}^{v,P}) \subset \text{Int } \mathcal{C}_{(x',y')}^{v,P}$$

and $\|DH_{(x,y)}^{-1}(\xi, \eta)\| \geq \frac{1}{|a|}\|(\xi, \eta)\|$ for $(\xi, \eta) \in \mathcal{C}_{(x,y)}^v$.

Proof. Let $(\xi, \eta) \in \mathcal{C}_{(x,y)}^v$ and denote by $(\xi', \eta') = DH_{(x,y)}^{-1}(\xi, \eta)$. Since

$$(x', y') = \left(\frac{y}{a}, \frac{x - p(y/a) - a^2 w}{a} \right) \quad \text{and} \quad DH_{(x,y)}^{-1} = \begin{bmatrix} 0 & \frac{1}{a} \\ \frac{1}{a} & -\frac{2y}{a^3} \end{bmatrix},$$

we can compute $\xi' = \frac{1}{a}\eta$ and $\eta' = \frac{1}{a}\left(\xi - \frac{2x'}{a}\eta\right)$. The vector (ξ, η) belongs to the vertical cone, so $\mu_U(x, \xi) = \rho_U(x)|\xi| < \tau|\eta|$. This implies

$$|\xi| < \frac{\tau}{m_1}|\eta|. \quad (44)$$

We can evaluate

$$\mu_U(x', \xi') = \rho_U(x')\frac{|\eta|}{|a|} \leq \frac{m_2}{|a|}|\eta|. \quad (45)$$

Next, we compute using inequality 44

$$|\eta'| > \frac{1}{|a|}\left|\xi - \frac{2x'}{a}\eta\right| > \frac{1}{|a|}\left(\frac{|2x'|}{|a|} - \frac{\tau}{m_1}\right)|\eta|. \quad (46)$$

The point x' belongs to the set U' . The set U' does not contain a neighborhood of the critical point 0 of the polynomial p . Hence there exists a lower bound $r_1 > 0$ such that $r_1 < |2x'|$. Choose a small, so $\frac{r_1}{|a|} - \frac{\tau}{m_1} > \max\left\{\frac{2m_2}{\tau}, 1\right\}$. By combining Equations 45 and 46 we get $\mu_U(x', \xi') < \frac{\tau}{2}|\eta|$. Therefore we have shown the invariance of vertical cones $DH_{(x,y)}^{-1}\left(\mathcal{C}_{(x,y)}^{v,P}\right) \subset \text{Int } \mathcal{C}_{(x',y')}^{v,P}$.

Inequality 46 shows that $|\eta'| > \frac{1}{|a|}|\eta|$, so DH^{-1} is expanding in the vertical cones. Since $\|(\xi, \eta)\| = \max(\mu_U(x, \xi), |\eta|) = |\eta|$ and $\|(\xi', \eta')\| = \max(\mu_U(x', \xi'), |\eta'|) = |\eta'|$, we obtain $\|(\xi', \eta')\| > \frac{1}{|a|}\|(\xi, \eta)\|$, as claimed. \square

Remark 9.6. The scalar $0 < \tau < 1$ in the definition of the vertical cone will typically be chosen less than $\left(\frac{\rho}{2}\right)^{2q}$, so that on a neighborhood of the boundary of B , the vertical cones $\mathcal{C}_{(x,y)}^{v,P}$ from Definition 9.4 are contained in the interior of the pull-back cones $\mathcal{C}_{(x,y)}^{v,B}$ from Definition 9.3. In this way we can assure that $DH_{(x,y)}^{-1}\left(\mathcal{C}_{(x,y)}^{v,P}\right) \subset \text{Int } \mathcal{C}_{(x',y')}^{v,B}$.

To fully show the invariance of the two types of vertical cones under DH^{-1} , we have one more case to cover.

Proposition 9.7. *Let $(x', y') \in B'$ and $(x, y) \in B$ such that $H(x', y') = (x, y)$. Then*

$$DH_{(x,y)}^{-1}\left(\mathcal{C}_{(x,y)}^{v,B}\right) \subset \text{Int } \mathcal{C}_{(x',y')}^{v,P}.$$

Proof. Consider $(\xi, \eta) \in \mathcal{C}_{(x,y)}^{v,B}$ and $(\xi', \eta') = DH_{(x,y)}^{-1}(\xi, \eta)$. We have to show that $\mu_U(x', \xi') < \tau|\eta'|$. Let $(\tilde{x}, \tilde{y}) = \phi_a(x, y)$ and $(\tilde{\xi}, \tilde{\eta}) = D\phi_a|_{(x,y)}(\xi, \eta)$. By Definition 9.3, the vector (ξ, η) belongs to the vertical cone $\mathcal{C}_{(x,y)}^{v,B}$ if and only if $|\tilde{\xi}| < |\tilde{x}|^{2q}|\tilde{\eta}|$. The change of coordinate $\phi_a(x, y)$ is holomorphic with respect to a and it is $\mathcal{O}(a)$ close to $(\phi(x), y)$, therefore there exists $\kappa_\phi > 0$ such that when a is small we have $|\xi| < \kappa_\phi|\eta|$.

By using the same computations as in Proposition 9.5 we obtain $\eta' = \frac{1}{a}\left(\xi - \frac{2x'}{a}\eta\right)$ and $\xi' = \frac{1}{a}\eta$ and we have estimates analogous to relations 45 and 46:

$$\mu_U(x', \xi') = \rho_U(x')|\xi'| < \frac{m_2}{|a|}|\eta| \quad (47)$$

$$|\eta'| > \frac{1}{|a|}\left(\frac{|2x'|}{|a|}|\eta| - |\xi|\right) > \frac{1}{|a|}\left(\frac{r_1}{|a|} - \kappa_\phi\right)|\eta|. \quad (48)$$

Therefore, to show that $\mu_U(x', \xi') < \tau|\eta'|$ all we need to know is that $m_2 < \tau \left(\frac{r_1}{|a|} - \kappa_\phi \right)$, which is obviously true when a is small. \square

Proposition 9.8 (Horizontal cones). *Let (x, y) and (x', y') in $(U' - \mathbb{D}_{\rho''}(q_0)) \times \mathbb{D}_r$ such that $H(x, y) = (x', y')$. Then*

$$DH_{(x,y)} \left(\mathcal{C}_{(x,y)}^{h,P} \right) \subset \text{Int } \mathcal{C}_{(x',y')}^{h,P}$$

and $\|DH_{(x,y)}(\xi, \eta)\| \geq k \cdot \|(\xi, \eta)\|$ for $(\xi, \eta) \in \mathcal{C}_{(x,y)}^{h,P}$, where $k > 1$ is a constant that depends on the polynomial p which has a parabolic fixed point at q_0 .

Proof. Let $(\xi, \eta) \in \mathcal{C}_{(x,y)}^{h,P}$ and $(\xi', \eta') = DH_{(x,y)}(\xi, \eta)$. We show that $(\xi', \eta') \in \text{Int } \mathcal{C}_{(x',y')}^{h,P}$ and $\mu_U(x', \xi') > k \cdot \mu_U(x, \xi)$. Since

$$(x', y') = (p(x) + a^2w + ay, ax) \quad \text{and} \quad DH_{(x,y)} = \begin{bmatrix} 2x & a \\ a & 0 \end{bmatrix},$$

we can compute $\xi' = 2x\xi + a\eta$ and $\eta' = a\xi$. The vector (ξ, η) belongs to the horizontal cone at (x, y) , so

$$|\eta| < \mu_U(x, \xi) = \rho_U(x)|\xi| < m_2|\xi|. \quad (49)$$

We evaluate

$$\mu_U(x', \xi') = \rho_U(p(x) + a^2w + ay) |2x\xi + a\eta|. \quad (50)$$

The density function ρ_U of the Poincaré metric is bounded above and below on the set $U'' := U' - \mathbb{D}_{\rho''}(q_0)$ since we removed a disk around the fixed point q_0 , where the boundaries of U' and U touch. Since ρ_U is C^∞ -smooth on U'' , its derivative ρ'_U is also bounded. There exists a constant $c_{U''} > 0$ such that

$$\begin{aligned} \frac{|\rho_U(p(x) + a^2w + ay) - \rho_U(p(x))|}{|a|} &\leq |aw + y| \cdot \sup_{U''} \rho'_U \cdot \frac{\rho_U(p(x))}{\inf_{U''} \rho_U} \\ &< c_{U''} \cdot \rho_U(p(x)). \end{aligned} \quad (51)$$

By 43, the polynomial p is expanding on $U' - \mathbb{D}_{\rho''}(q_0)$ with respect to the Poincaré metric μ_U , that is there exists a constant $k_0 > 1$ such that

$$\rho_U(p(x)) |p'(x)\xi| > k_0 \cdot \rho_U(x)|\xi|. \quad (52)$$

The set U' avoids a uniform neighborhood of the critical point 0, so there exists as before r_1 such that $0 < r_1 < |2x|$ for any $x \in U'$. We now turn back to relation 50. Using 51, 52 and 49 one gets

$$\begin{aligned} \mu_U(x', \xi') &> (1 - c_{U''}|a|) \cdot \rho_U(p(x)) |2x\xi| \cdot \frac{|2x\xi + a\eta|}{|2x\xi|} \\ &> (1 - c_{U''}|a|) \cdot k_0 \cdot \rho_U(x)|\xi| \cdot \left(1 - |a| \frac{|\eta|}{|2x||\xi|} \right) \\ &> k_0 \cdot (1 - c_{U''}|a|) \cdot \left(1 - |a| \frac{m_2}{r_1} \right) \cdot \rho_U(x)|\xi|. \end{aligned} \quad (53)$$

The constant k_0 is strictly bigger than 1. The factors $1 - c|a|$ and $1 - |a|\frac{m_2}{r_1}$ can be made arbitrarily close to 1 by reducing $|a| < \delta$. Let δ be sufficiently small so that

$$k := k_0 \cdot (1 - c_U \delta) \cdot \left(1 - \delta \frac{m_2}{r_1}\right) > 1. \quad (54)$$

When $|a| < \delta$, relation 53 gives

$$\mu_U(x', \xi') > k \cdot \rho_U(x) |\xi| = k \cdot \mu_U(x, \xi) \quad (55)$$

which proves that DH expands in the horizontal cones. Also from relation 53 we infer that

$$|a| \cdot \mu_U(x', \xi') > k \cdot \rho_U(x) |a\xi| > k \cdot m_1 \cdot |\eta'| \quad (56)$$

which proves that $DH_{(x,y)} \left(\mathcal{C}_{(x,y)}^{h,P} \right) \subset \text{Int } \mathcal{C}_{(x',y')}^{h,P}$, so the horizontal cones are invariant under DH for a small. \square

If both types of horizontal cones are defined at some point $(x, y) \in V$, we cannot say in general that one is contained in the other, i.e. $\mathcal{C}_{(x,y)}^{h,P} \subset \mathcal{C}_{(x,y)}^{h,B}$ or $\mathcal{C}_{(x,y)}^{h,B} \subset \mathcal{C}_{(x,y)}^{h,P}$. However, since the derivative of the Hénon map contracts the vertical component of tangent vectors by a factor of a , we can assume that $DH_{(x,y)} \left(\mathcal{C}_{(x,y)}^{h,P} \right) \subset \text{Int } \mathcal{C}_{(x',y')}^{h,B}$ and $DH_{(x,y)} \left(\mathcal{C}_{(x,y)}^{h,B} \right) \subset \text{Int } \mathcal{C}_{(x',y')}^{h,P}$ whenever both types of cones are defined at (x, y) and/or at $(x', y') = H(x, y)$.

10. DISTANCE BETWEEN VERTICAL-LIKE CURVES

In this section we work entirely in the normalized coordinates from Theorem 6.2. The notion of vertical-like curves translates as follows

Definition 10.1. We will call an analytic curve $\gamma \subset \mathbb{D}_\rho \times \mathbb{D}_r$ *vertical-like* if γ is the graph of an analytic function $\phi : \mathbb{D}_r \rightarrow \mathbb{D}_\rho$, and for all points (x, y) on γ , the tangent vectors (ξ, η) to γ at (x, y) belong to the vertical cone $\mathcal{C}_{(x,y)}^v$ from Definition 9.1.

Let us now consider two vertical-like curves in the same repelling sector of $\mathbb{D}_\rho \times \mathbb{D}_r$, that are entirely contained in the escaping set U^+ . Denote these vertical curves

$$f_1(z) = (\varphi_1(z), z) \text{ and } f_2(z) = (\varphi_2(z), z).$$

Let $g_1(\mathbb{D}_r)$ be the image under \tilde{H}^{-1} of $f_1(\mathbb{D}_r)$, contained inside $\mathbb{D}_\rho \times \mathbb{D}_r$. More precisely,

$$\tilde{H}^{-1}(f_1(\mathbb{D}_r)) \cap (\mathbb{D}_\rho \times \mathbb{D}_r),$$

is a vertical-like fiber, that we can describe as the graph of an analytic function

$$g_1(z) = (\varphi'(z), z), \text{ where } \varphi' : \mathbb{D}_r \rightarrow \mathbb{D}_\rho.$$

Similarly, let $g_2(\mathbb{D}_r)$ be $\tilde{H}^{-1}(f_2(\mathbb{D}_r)) \cap (\mathbb{D}_\rho \times \mathbb{D}_r)$ reparametrized by the second coordinate $g_2(z) = (\varphi''(z), z)$. Notice that $g_1(\mathbb{D}_r)$ and $g_2(\mathbb{D}_r)$ are vertical-like curves (by Proposition 9.2), both contained in some other repelling sector of $\mathbb{D}_\rho \times \mathbb{D}_r$ and in U^+ .

Much like in the hyperbolic setting, we would like to show that \tilde{H} expands the horizontal distance between vertical-like curves. We will measure the horizontal distance with respect to the standard Euclidean metric on $\mathbb{D}_\rho \times \mathbb{D}_r$. Define

$$d(f_1, f_2) = \|\varphi_1 - \varphi_2\| = \sup_{z \in \mathbb{D}_r} |\varphi_1(z) - \varphi_2(z)|$$

Notice that this distance between vertical-like curves is just the distance between the parametrizing functions φ_1 and φ_2 with respect to the sup-norm.

Theorem 10.2. *Let $d(g_1, g_2)$ and $d(f_1, f_2)$ be the horizontal distance between the vertical-like curves g_1, g_2 and respectively f_1, f_2 . Then*

$$d(g_1, g_2) < Cd(f_1, f_2),$$

where $C = C(f_1, f_2) < 1$, so the normalized Hénon maps \tilde{H} expands strictly the distance between the vertical-like curves g_1 and g_2 .

Proof. Let $z \in \mathbb{D}_r$ be arbitrarily chosen and denote by $x' = \varphi'_1(z)$, and $x'' = \varphi''_1(z)$. The points (x', z) and (x'', z) lie on the vertical-like curves g_1 and g_2 .

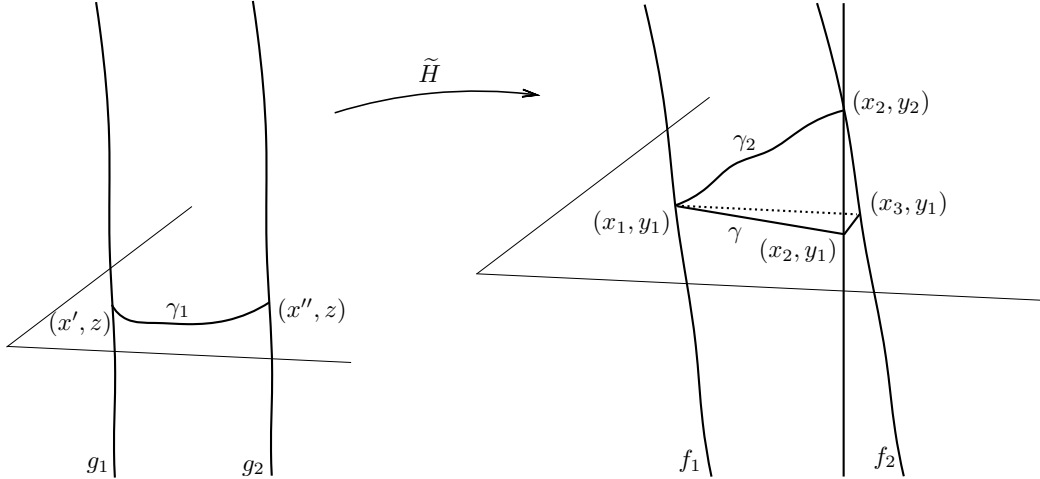


Figure 7. Fibers g_1, g_2 and their image fibers f_1, f_2 under \tilde{H} .

Let $(x_1, y_1) = \tilde{H}(x', z)$ and $(x_2, y_2) = \tilde{H}(x'', z)$ be the corresponding points on f_1 and f_2 . Let $(x_3, y_1) = (\varphi_2(y_2), y_2)$ be the point of intersection of the curve f_2 with the horizontal plane $\mathbb{C} \times \{y_1\}$. Suppose without loss of generality that $|x_2| \leq |x_1|$. Let

$$\mathcal{I}(x_1, x_2) := \int_0^1 |tx_1 + (1-t)x_2|^q dt.$$

Lemma 10.3 (Step 1). *We have*

$$|x' - x''| < \left(1 - \frac{\epsilon_1}{2M_1^q} \mathcal{I}(x_1, x_2)\right) |x_1 - x_2|,$$

where the constant M_1 is independent of a . The constant ϵ_1 is given in Equation 23.

Proof. Choose a straight line in the $\mathbb{C} \times \{y_1\}$ plane,

$$\gamma(t) = (x_\gamma(t), y_1), \quad \text{where } x_\gamma(t) = tx_1 + (1-t)x_2 \text{ and } t \in [0, 1],$$

connecting the points (x_1, y_1) and (x_2, y_1) . There exists a horizontal curve

$$\gamma_1(t) : [0, 1] \rightarrow \mathbb{D}_\rho \times \{z\}, \quad \gamma_1(t) = (x_{\gamma_1}(t), z),$$

connecting the points $(x', z) = \gamma_1(0)$ and $(x'', z) = \gamma_1(1)$ and such that the projection of the curve $\gamma_2(t) := \tilde{H}(\gamma_1(t))$ on the plane $\mathbb{C} \times \{y_1\}$ is exactly the straight line $\gamma(t)$. Formally, if we define $pr : \mathbb{D}_\rho \times \mathbb{D}_r \rightarrow \mathbb{D}_\rho \times \{y_1\}$, $pr(x, y) = (x, y_1)$, then $pr(\gamma_2(t)) = \gamma(t)$. By Lemma 6.6, we know that $D\tilde{H}$ expands the horizontal length of vectors in W^- , so

$$|\gamma'(t)| > C(x_{\gamma_1}(t))|\gamma_1'(t)|.$$

We will compare the length of the curve γ_1 with the length of γ . Note that $\gamma(t)$ is a just a horizontal line segment, hence $|\gamma'(t)| = |x_1 - x_2|$, for all $t \in [0, 1]$ and $l(\gamma) = |x_1 - x_2|$.

$$l(\gamma_1) = \int_0^1 |\gamma_1'(t)| dt < \int_0^1 \frac{1}{C(x_{\gamma_1}(t))} |\gamma'(t)| dt = |x_1 - x_2| \int_0^1 \frac{1}{C(x_{\gamma_1}(t))} dt \quad (57)$$

Recall from Equation 30 that $C(x) := |1 + (q+1)x^q| - m|x|^{2q} \geq 1 + \epsilon_1|x|^q$. We have $C(x) > 1$ for all $(x, y) \in W^-$. Since $|x| < 1$, we also have that

$$\frac{1}{C(x)} \leq \frac{1}{1 + \epsilon_1|x|^q} \leq 1 - \frac{\epsilon_1}{2}|x|^q.$$

Recall also that for any $t \in [0, 1]$

$$x_\gamma(t) = \lambda x_{\gamma_1}(t) (1 + x_{\gamma_1}(t)^q + g_a(x_{\gamma_1}(t), z)/x_{\gamma_1}(t))$$

where $|1 + x_{\gamma_1}(t)^q + g_a(x_{\gamma_1}(t), z)/x_{\gamma_1}(t)| < M_1$, as in Equation 41. By combining these estimates we obtain

$$\begin{aligned} \int_0^1 \frac{1}{C(x_{\gamma_1}(t))} dt &< 1 - \frac{\epsilon_1}{2} \int_0^1 |x_{\gamma_1}(t)|^q dt < 1 - \frac{\epsilon_1}{2M_1^q} \int_0^1 |x_\gamma(t)|^q dt \\ &= 1 - \frac{\epsilon_1}{2M_1^q} \int_0^1 |tx_1 + (1-t)x_2|^q dt. \end{aligned}$$

Using that $|x' - x''| \leq l(\gamma_1)$ and Equation 57 we get the desired inequality. \square

Lemma 10.4 (Technical estimate). *Let $q \geq 1$ be a natural number and $x_1, x_2 \in \mathbb{C}$ be two complex numbers, with $|x_2| \leq |x_1|$. Then $|x_1|^q \leq 2(q+1)\mathcal{I}(x_1, x_2)$.*

Proof. If $x_1 = 0$ then $x_2 = 0$ and we have equality. Otherwise, set $x = x_2/x_1$. Then $|x| \leq 1$ and we need to show that

$$\frac{1}{2(q+1)} \leq \int_0^1 \left| t \frac{x_2}{x_1} + (1-t) \right|^q dt = \int_0^1 |tx + (1-t)|^q dt.$$

For any $t \in [0, 1]$ we have $|tx + (1-t)| \geq |t|x - (1-t)| = |t(1+|x|) - 1|$. Let $u = t(1+|x|) - 1$. Then $du = (1+|x|)dt$ and

$$\int_0^1 |t(1+|x|) - 1|^q dt = \frac{1}{|x|+1} \int_{-1}^{|x|} |u|^q du = \frac{1}{|x|+1} \frac{|x|^{q+1} + 1}{q+1} > \frac{1}{2(q+1)},$$

since $0 \leq |x| \leq 1$. \square

Suppose for now that a is small enough such that $N_a < \frac{1}{8q}$. This is similar to what we previously required for N_a . In Equation 59 we will impose another bound for N_a .

Lemma 10.5 (Step 2). $|x_2 - x_3| < 4(q+1)N_a\mathcal{I}(x_1, x_2)|x_1 - x_2|$.

Proof. The geometric intuition behind the inequality is that the curve γ_2 connecting (x_1, y_1) and (x_2, y_2) becomes horizontal as $a \rightarrow 0$, while the fibers f_1 and f_2 become vertical. The rigorous proof is outlined below. From the proof of Proposition 6.8 it follows that $|y_1 - y_2| < N_a|x_1 - x_2|$, where $N_a \rightarrow 0$ as $a \rightarrow 0$. The curve

$$t \rightarrow (\varphi_2(ty_1 + (1-t)y_2), ty_1 + (1-t)y_2), \quad t \in [0, 1]$$

is vertical-like so in particular the horizontal distance is smaller than the vertical distance and

$$|\varphi_2(ty_1 + (1-t)y_2) - \varphi_2(y_2)| < |ty_1 + (1-t)y_2 - y_2| = t|y_1 - y_2|,$$

for $t \in [0, 1]$. Using $\varphi_2(y_2) = x_2$ this gives

$$|\varphi_2(ty_1 + (1-t)y_2)| < |x_2| + t|y_1 - y_2| < |x_2| + tN_a|x_1 - x_2|.$$

Hence

$$\begin{aligned} |x_2 - x_3| &\leq \int_0^1 \left| \frac{\partial}{\partial t} \varphi_2(ty_1 + (1-t)y_2) \right| dt \leq \int_0^1 |y_1 - y_2| |\varphi_2(ty_1 + (1-t)y_2)|^{2q} dt \\ &\leq |y_1 - y_2| (|x_2| + N_a|x_1 - x_2|)^{2q} \leq N_a|x_1 - x_2| (|x_2| + N_a|x_1 - x_2|)^{2q}. \end{aligned}$$

Suppose without loss of generality that $|x_2| \leq |x_1|$ and $|x_1| < 1$. From the technical estimate Lemma 10.4 we get

$$\begin{aligned} |x_2 - x_3| &< N_a|x_1 - x_2|(1 + 2N_a)^{2q}|x_1|^{2q} \\ &< N_a|x_1 - x_2|(1 + 2N_a)^{2q} \cdot 2(q+1)\mathcal{I}(x_1, x_2). \end{aligned}$$

Since $N_a < \frac{1}{8q}$, we can use the following estimate

$$2(q+1)(1 + 2N_a)^{2q} < 2(q+1) \left(1 + \frac{1}{4q}\right)^{2q} < 4(q+1)$$

to get $|x_2 - x_3| < 4(q+1)N_a\mathcal{I}(x_1, x_2)|x_1 - x_2|$. \square

We now return to the proof of Theorem 10.2. We can use the triangle inequality in the $\mathbb{D}_\rho \times \{y_1\}$ disk to connect $|x_1 - x_2|$ to the distance between the curves f_1 and f_2

$$|x_1 - x_2| - |x_2 - x_3| \leq |x_1 - x_3| \leq d(f_1, f_2).$$

In Step 2 we showed that $|x_2 - x_3| < 4(q+1)N_a\mathcal{I}(x_1, x_2)|x_1 - x_2|$, so

$$(1 - 4(q+1)N_a\mathcal{I}(x_1, x_2))|x_1 - x_2| < d(f_1, f_2).$$

Using the bound on $|x_1 - x_2|$ from Step 1, we get that

$$|x' - x''| < \frac{1 - \frac{\epsilon_1}{2M_1^q}\mathcal{I}(x_1, x_2)}{1 - 4(q+1)N_a\mathcal{I}(x_1, x_2)} d(f_1, f_2),$$

where the quantity

$$C = \frac{1 - \frac{\epsilon_1}{2M_1^q} \mathcal{I}(x_1, x_2)}{1 - 4(q+1)N_a \mathcal{I}(x_1, x_2)} < 1, \quad (58)$$

for a small enough. Indeed, the constants q, ϵ_1 and M_1 are independent of the parameter a whereas $N_a \rightarrow 0$ as $a \rightarrow 0$, so it can be made small enough so that

$$N_a < \frac{\epsilon_1}{8(q+1)M_1^q}. \quad (59)$$

The right hand side is a fixed constant, but this bound is not optimized. We get that

$$d(g_1, g_2) \leq Cd(f_1, f_2),$$

where $C < 1$ depends on the distance between the curves and the y -axis. This dependency is hidden in $\mathcal{I}(x_1, x_2)$. \square **of Theorem 10.2**

11. THE FIXED POINT OF A WEAKLY CONTRACTING OPERATOR

In this section, we construct a function space \mathcal{F} and a graph transform operator $F : \mathcal{F} \rightarrow \mathcal{F}$. We endow the space \mathcal{F} with a metric induced by μ on the set V and show that the operator F is strictly (but not strongly) contracting. The key ingredients will be the invariance of vertical cones constructed in Section 9 under DH^{-1} and the weak expansion of DH in the horizontal cones. We use a generalization of the Banach fixed point theorem, due to Browder, to claim the existence of a unique fixed point f^* of F .

Lemma 11.1. *Let $(x, y) \in V \cap \overline{U^+}$ and $(x', y') = H^{-1}(x, y)$. If $|y'| < r$ then $(x', y') \in V$.*

Proof. The point (x', y') is in $U^+ \cup J^+$ hence it cannot lie in the sets that have been removed from $\mathbb{D}_r \times \mathbb{D}_r$ when constructing the set V as they belong to the interior of K^+ , as shown in Lemmas 7.1 and 6.6.

If (x, y) is in V , then it belongs to W_B^- , $W_{B'}^-$ or $(U' - \mathbb{D}_{\rho'}(q_0)) \times \mathbb{D}_r$. In the first case, if (x, y) belongs to the repelling sectors W_B^- , then $(x', y') \in W_B^- \cup W_{B'}^- \subset V$. In the last two cases, if (x, y) belongs to $W_{B'}^-$ or $(U' - \mathbb{D}_{\rho'}(q_0)) \times \mathbb{D}_r$, then for a chosen small enough, we can assume that the disk of radius $2r|a|$ around x is contained in U . The point $(x', y') = \frac{1}{a}(y, x - p(y/a) - a^2w)$ belongs to V because $|y'| < r$ (by hypothesis) and $y/a \in U'$. We can use the inequality $|x - p(y/a) - a^2w| < r|a|$ to show that $y/a \in U'$. Indeed, since a is chosen small enough so that the disk of radius $r|a| + |w||a|^2 < 2r|a|$ around x is still in U , it follows that $p(y/a) \in U$, hence $y/a \in U'$. Then $(x', y') \in U' \times \mathbb{D}_r$ and $(x', y') \in U^+ \cup J^+$, hence (x', y') belongs to V . \square

Definition 11.2. Let $L = \{(f(z), z), z \in \mathbb{D}_r\} \subset V$, be the graph of an analytic function $f : \mathbb{D}_r \rightarrow \mathbb{D}_r$. The analytic curve $L \subset V$ is *vertical-like*, if the following conditions are met. Choose $(x, y) \in L$ and (ξ, η) a tangent vector to L at (x, y) . If $(x, y) \in B$, then (ξ, η) belongs to the pull-back vertical cone $\mathcal{C}_{(x,y)}^{v,B}$ described in Definition 9.3 using Definition 9.1. If (x, y) is outside B'' then (ξ, η) belongs to the vertical cone $\mathcal{C}_{(x,y)}^{v,P}$ described in Definition 9.4.

Lemma 11.3. *Let L be a vertical-like curve in $V \cap (U^+ \cup J^+)$. Then $H^{-1}(L) \cap V$ is the union of two vertical-like curves L_1 and L_2 .*

Proof. Since the curve L is vertical-like, it is the graph of a holomorphic function $f : \mathbb{D}_r \rightarrow \mathbb{D}_r$, with $|f'(z)| < 1$. Hence $L = \{(f(z), z), z \in \mathbb{D}_r\}$. Then

$$H^{-1}(L) = \{(z/a, (f(z) - p(z/a) - a^2w)/a), z \in \mathbb{D}_r\}$$

is an analytic curve whose horizontal folding points cannot belong to V . Suppose there is a folding point inside V . By construction of V , the first coordinate z/a of the folding point must be bounded away from 0 (independent of a). It follows that the equation $f'(z) - 2z/a^2 = 0$ cannot have solutions inside \mathbb{D}_r , as $(2/a) \cdot (z/a)$ gets arbitrarily large when a is small enough, whereas $f'(z)$ remains bounded, because the curve is vertical-like.

Therefore the degree of the projection of $H^{-1}(L) \cap V$ on the second coordinate is constant. It is easy to see that the degree is 2, by looking at the number of intersections of $H^{-1}(L)$ with the x -axis. The curve L is vertical-like in V , hence it intersects $H(x\text{-axis})$ in exactly two points. Then $H^{-1}(L)$ intersects the x -axis in two points.

Thus $H^{-1}(L) \cap \mathbb{C} \times \mathbb{D}_r$ is a union of two analytic curves L_1 and L_2 . By Lemma 11.1, L_1 and L_2 are contained in V , hence $H^{-1}(L) \cap V$ is the union of two analytic curves L_1 and L_2 .

Let us make one more remark about L_1 and L_2 . Since L is a vertical like disk in U^+ , its projection on the first coordinate is almost constant and bounded away from 0, the critical point of p . Let Δ be the image of the projection on the first coordinate. There are two holomorphic branches p_1 and p_2 of p^{-1} defined on Δ . Let now $(f(z), z)$ be any point of L such that $H^{-1}(f(z), z) \in V$. By Lemma 11.1 it suffices to check that the condition $|(f(z) - p(z/a) - a^2w)/a| < r$ is met. This condition means exactly that z/a is $\mathcal{O}(a)$ close to either $p_1(f(z))$ or $p_2(f(z))$. The curves L_1 and L_2 correspond to different choices of the branch of p^{-1} .

By Lemmas 9.2 and 9.5, the curves L_1 and L_2 are vertical-like, so they are graphs of functions on \mathbb{D}_r . The projection $pr_2 : L_1 \rightarrow \mathbb{D}_r$, $pr_2(x, y) = y$ is a degree one covering map, hence by the Implicit Function Theorem, L_1 is the graph of a holomorphic function $x = \phi(y)$. The map ϕ must also be injective, as the projection on the first coordinate $pr_1 : L_1 \rightarrow U'$, $pr_1(x, y) = x$ is injective. \square

Choose $R > 2$ as in the construction of the neighborhood U of the Julia set J_p from Equation 36 and define the sequence of equipotentials $\gamma_n : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$ of the parabolic polynomial p ,

$$\gamma_{n+1}(t) = \Phi_p \left(R^{1/2^{n+1}} e^{2\pi i t} \right) = p^{-1}(\gamma_n(2t)), \quad (60)$$

where Φ_p is the inverse Böttcher isomorphism of p 1. Note that $\gamma_{-1}(\mathbb{R}/\mathbb{Z}) \subset \partial U$ and $\gamma_0(\mathbb{R}/\mathbb{Z}) \subset \partial U'$.

Definition 11.4. Let $f_0 : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow \partial V$ be the function $f_0(t, z) = (\gamma_0(t), z)$. The image of f_0 is the outer boundary of V and it is contained in the escaping set U^+ .

We will construct a sequence of pull-backs of the map f_0 under the Hénon map. This choice of f_0 will simplify the computations from Section 12 where we establish the conjugacy of the Hénon map to a model map. For the definition of the function space we could start with any function $f_0 : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow U^+ \cap V$, $f_0(t, z) = (\phi_t(z), z)$, continuous with respect to t and analytic with respect to z such that $\phi_t(\mathbb{D}_r)$ is a vertical-like disk and $t \mapsto \phi_t(0)$ is homotopic to $\gamma_0(t)$.

Definition 11.5. Consider the space of functions:

$$\mathcal{F} = \{f_n : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow V \mid f_0(t, z) = (\gamma_0(t), z), f_n(t, z) = F \circ f_{n-1}(t, z) \text{ for } n \geq 1\},$$

where the graph transform $F : \mathcal{F} \rightarrow \mathcal{F}$ is defined as

$$F(f) = \tilde{f},$$

where $\tilde{f}|_{t \times \mathbb{D}_r}$ is the conformal map of the component of $H^{-1}(f(2t \times \mathbb{D}_r)) \cap V$ “homotopic to” $f_0(t \times \mathbb{D}_r)$, normalized via the Implicit Function Theorem (the projection on the second coordinate).

On the function space \mathcal{F} we consider the metric

$$d(f, g) = \sup_{t \in \mathbb{S}^1} \sup_{z \in \mathbb{D}_r} d_\mu(f(t, z), g(t, z)).$$

where $d_\mu(f(t, z), g(t, z))$ is the infimum of the length of horizontal rectifiable paths $\tau : [0, 1] \rightarrow V$ with $\tau(0) = f(t, z)$ and $\tau(1) = g(t, z)$. The length is measured with respect to μ , which is defined in Section 8. Note that d_μ is a metric.

We will begin by describing the function space \mathcal{F} and showing how the function f_1 is constructed. Proposition 11.7 explains the construction of the graph-transform F and describes the main properties of the functions from the space \mathcal{F} .

For any fixed $t \in \mathbb{S}^1$, the set $f_0(2t \times \mathbb{D}_r)$ is a vertical disk. By Lemma 11.3, the preimage $H^{-1}(f_0(2t \times \mathbb{D}_r)) \cap V$ is a disjoint union of two vertical-like disks (as shown in Figure 8) that we would like to label as t and $t + 1/2$.

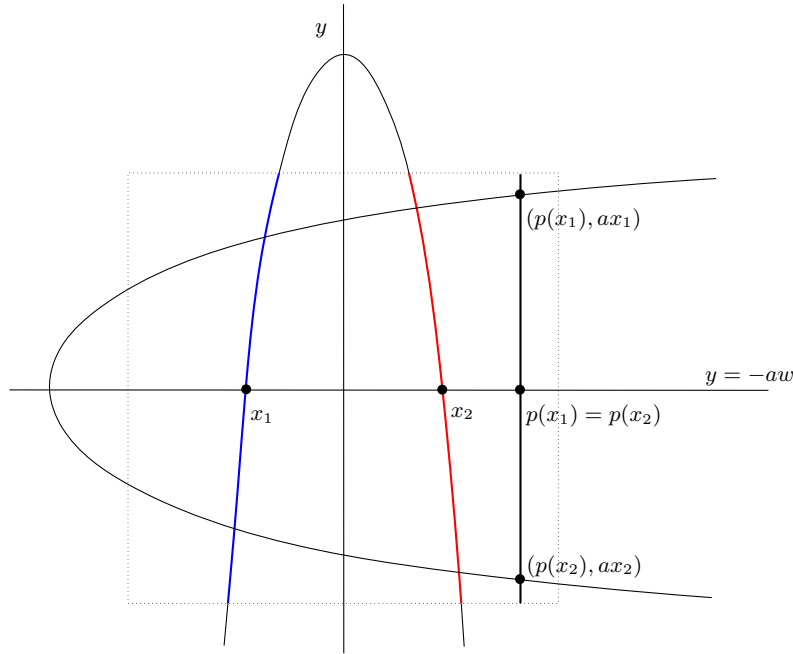


Figure 8. The preimage of a fiber of f_0 in the neighborhood V .

The preimage $H^{-1}(\gamma_0(2t), z) = (z/a, (\gamma_0(2t) - p(z/a) - a^2w)/a)$ belongs to V if and only if the second component belongs to \mathbb{D}_r . The inequality $|\gamma_0(2t) - p(z/a) - a^2w| < r|a|$ implies that the first coordinate z/a is $\mathcal{O}(a)$ close to one of the two preimages of $\gamma_0(2t)$ under the polynomial p .

Denote by C_t and $C_{t+1/2}$ the components of $H^{-1} \circ f_0(2t \times \mathbb{D}_r) \cap V$ that cross the horizontal axis $y = -aw$ at $(\gamma_1(t), -aw)$, and respectively at $(\gamma_1(t + 1/2), -aw)$.

Notice that $pr_2 : C_t \rightarrow \mathbb{D}_r$, $pr_2(x, z) = z$ is a degree one covering map, hence, by the Implicit Function Theorem, C_t is the graph of a holomorphic function $x = \phi_t(z)$. Let us define a new function $f_1 : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow V$ as $f_1(t, z) := (\phi_t(z), z)$.

Remark 11.6. f_1 is homotopic to f_0 by construction, since $\gamma_1(t)$ and $p^{-1}(\gamma_0(2t))$ are homotopic. Moreover, since a is small, $f_1(\mathbb{S}^1 \times \mathbb{D}_r)$ and $f_0(\mathbb{S}^1 \times \mathbb{D}_r)$ are disjoint.

Proposition 11.7. *The map $F : \mathcal{F} \rightarrow \mathcal{F}$ is well defined. Choose any function $f \in \mathcal{F}$. For any $t \in \mathbb{S}^1$, $f(t \times \mathbb{D}_r)$ is a vertical-like disk parametrized by the second coordinate. There exists $\varphi_t : \mathbb{D}_r \rightarrow \mathbb{D}_r$ analytic with respect to z and a and continuous with respect to t such that $f(t, z) = (\varphi_t(z), z)$.*

Proof. We have $F \circ f_0(t, z) = f_1(t, z)$. Assume by induction that the functions $f_{n-1}, f_n : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow V$, $f_n(t, z) = (\varphi_t^n(z), z)$, $f_{n-1}(t, z) = (\varphi_t^{n-1}(z), z)$ have been constructed for $n \geq 1$ and let us show how to define f_{n+1} .

For each $t \in \mathbb{S}^1$, $H^{-1}(f_n(2t \times \mathbb{D}_r)) \cap V$ is a union of two vertical like disks in U^+ , C_t and $C_{t+1/2}$. A choice of labeling is involved and we will first explain how this is done. Intuitively we would like to label by t the disk which is closer to $f_n(t \times \mathbb{D}_r)$ and by $t + 1/2$ the disk which is closer to $f_n((t + 1/2) \times \mathbb{D}_r)$. Let $(x_1, -aw)$ and $(x_2, -aw)$ be the two intersection points of $H^{-1}(f_n(2t \times \mathbb{D}_r))$ with the axis $y = -aw$. The points $(p(x_1), ax_1)$ and $(p(x_2), ax_2)$ belong to $f_n(2t \times \mathbb{D}_r)$ and we would like to label them. The disk $f_{n-1}(2t \times \mathbb{D}_r)$ contains two labeled points $b_t^n = H(f_n(t, -aw))$ and $b_{t+1/2}^n = H(f_n(t + 1/2, -aw))$. Let p_1 and p_2 be two holomorphic branches of p such that $p_1 \circ p(\varphi_t^n(-aw)) = \varphi_t^n(-aw)$ and $p_2 \circ p(\varphi_{t+1/2}^n(-aw)) = \varphi_{t+1/2}^n(-aw)$. If $p_1 \circ p(x_1) = x_1$ then we label the point $(p(x_1), ax_1)$ as b_t^{n+1} and the component of $H^{-1}(f_n(2t \times \mathbb{D}_r)) \cap V$ that intersects the axis $y = -aw$ at $(x_1, -aw)$ as C_t . Otherwise, if $p_2 \circ p(x_1) = x_1$, we label it as $C_{t+1/2}$.

As before, the projection on the second coordinate $pr_2 : C_t \rightarrow \mathbb{D}_r$, $pr_2(x, z) = z$ is a degree one covering map, hence, by the Implicit Function Theorem, C_t is the graph of a holomorphic function $x = \varphi_t^{n+1}(z)$. Let then f_{n+1} be defined as

$$f_{n+1}(t, z) := (\varphi_t^{n+1}(z), z).$$

It is also easy to see that φ_t^{n+1} is injective, by the definition of H^{-1} . The function f_{n+1} is holomorphic with respect to z and a and continuous with respect to t . \square

Proposition 11.8. *The operator $F : \mathcal{F} \rightarrow \mathcal{F}$ is a strict contraction.*

$$d(F(f), F(g)) < d(f, g), \text{ for any } f, g \in \mathcal{F}.$$

The proof is an immediate consequence of the following proposition.

Proposition 11.9. *Let $f, g \in \mathcal{F}$ and $t \in \mathbb{S}^1$. Then*

$$d(F \circ f(t \times \mathbb{D}_r), F \circ g(t \times \mathbb{D}_r)) < C(f, g, t) \cdot d(f(2t \times \mathbb{D}_r), g(2t \times \mathbb{D}_r)),$$

where $C(f, g, t)$ is a contraction factor which depends on the fibers $f(t \times \mathbb{D}_r)$ and $g(t \times \mathbb{D}_r)$ and $0 \leq C(f, g, t) < 1$.

Proof. The delicate case is when the curves enter B'' and come close to $W_{loc}^s(\mathbf{q}_a)$. By Theorem 8.7, case (b), the derivative of the Hénon map still expands in the horizontal direction but the expansion factor goes to 1 as we approach the stable manifold. This case has already been carefully analyzed in Section 10.

The case where the curves are outside of a small neighborhood of the local stable manifold $W_{loc}^s(\mathbf{q}_a)$ can be treated as in the hyperbolic setting, because by Theorem 8.7 the derivative of the Hénon map expands in the horizontal direction with a fixed expansion factor, independent of a .

Suppose that the fibers $f(2t \times \mathbb{D}_r), g(2t \times \mathbb{D}_r), F \circ f(t \times \mathbb{D}_r), F \circ g(t \times \mathbb{D}_r)$ belong to $V - B = (U' - \mathbb{D}_{\rho'}(q_0)) \times \mathbb{D}_r$. We show that there exists a constant $C < 1$ such that

$$\sup_{z \in \mathbb{D}_r} d_\mu(F \circ f(t, z), F \circ g(t, z)) \leq C \sup_{z \in \mathbb{D}_r} d_\mu(f(2t, z), g(2t, z)). \quad (61)$$

Recall that $f(2t \times \mathbb{D}_r), g(2t \times \mathbb{D}_r), F \circ f(t \times \mathbb{D}_r)$ and $F \circ g(t \times \mathbb{D}_r)$ are vertical-like disks parametrized by the second coordinate. There exists conformal maps $\varphi_1, \varphi_2 : \mathbb{D}_r \rightarrow U'$ such that $F \circ f(t, z) = (\varphi_1(z), z)$ and $F \circ g(t, z) = (\varphi_2(z), z)$. Let z be any point in \mathbb{D}_r . Denote by $x' = \varphi_1(z)$, $x'' = \varphi_2(z)$. Then $(x_1, y_1) = H(x', z) = (p(x') + az + a^2w, ax')$ and $(x_2, y_2) = H(x'', z) = (p(x'') + az + a^2w, ax'')$. Finally, denote by (x_3, y_1) the intersection point of the vertical-like curve $g(2t \times \mathbb{D}_r)$ with the horizontal line $\mathbb{C} \times \{y_1\}$. The configuration is the same as in Figure 7 (fibers named f and g here correspond to f_1 and g_1 on the picture).

Let γ be any horizontal curve in V between the point (x_1, y_1) on the curve $f(2t \times \mathbb{D}_r)$ and the point (x_2, y_1) . There exists a horizontal curve γ_1 in V between the point (x', z) on the curve $F \circ f(t \times \mathbb{D}_r)$ and (x'', z) on the curve $F \circ g(t \times \mathbb{D}_r)$ such that $\gamma_2 = H(\gamma_1)$ is a curve inside the parabola $H(\mathbb{C} \times z)$ linking (x_1, y_1) to (x_2, y_2) such that its projection on the plane $\mathbb{C} \times y_1$ is exactly the curve γ . The curves γ, γ_1 and γ_2 are just local variables here.

The Hénon map expands the length of horizontal vectors, therefore by Definition 8.3 and Proposition 9.8 there exists $k > 1$ such that

$$\mu_P(\gamma_2(s), \gamma_2'(s)) > k\mu_P(\gamma_1(s), \gamma_1'(s)), \text{ for any } s \in [0, 1].$$

Notice also that the tangent vector $\gamma_2'(s)$ belongs to the horizontal cone at $\gamma_2(s)$, so in particular, μ_P is just the Poincaré metric of the projection on the first coordinate, so

$$\mu_P(\gamma_1(s), \gamma_1'(s)) = \mu_P(\gamma(s), \gamma'(s)) = \mu_U(pr_1(\gamma(s)), pr_1(\gamma'(s))).$$

After passing to the infimum after the length of all horizontal curves γ we conclude that

$$d_\mu((x_1, y_1), (x_2, y_1)) > kd_\mu((x', z), (x'', z)),$$

where the distance is measured with respect to the Poincaré metric of U .

Let $\psi : \mathbb{D}_r \rightarrow g(2t \times \mathbb{D}_r)$ be the conformal isomorphism which parametrizes the fiber. Then $\psi(ax') = (x_3, y_1)$ and $\psi(ax'') = (x_2, y_2)$. The fiber is vertical-like, therefore

$$d_\mu((x_2, y_1), (x_3, y_1)) \leq \tau |ax' - ax''|$$

The points x' and x'' are in $U' - \mathbb{D}_{\rho'}(q_0)$ which is compactly contained in U , so we have

$$m_1|x' - x''| \leq d_\mu((x', z), (x'', z)) \leq m_2|x' - x''|.$$

In conclusion we have $d_\mu((x_2, y_1), (x_3, y_1)) \leq |a| \frac{\tau}{m_1} d_\mu((x', z), (x'', z))$. By the triangle inequality it follows that

$$\begin{aligned} d_\mu((x_1, y_1), (x_2, y_1)) - d_\mu((x_2, y_1), (x_3, y_1)) &\leq d_\mu((x_1, y_1), (x_3, y_1)) \\ &\leq \sup_{z \in \mathbb{D}_r} d_\mu(f(2t, z), g(2t, z)). \end{aligned}$$

So by combining the previous inequalities we get

$$d_\mu((x', z), (x'', z)) \leq \frac{1}{(k - |a| \frac{\tau}{m_1})} \sup_{z \in \mathbb{D}_r} d_\mu(f(2t, z), g(2t, z))$$

The expansion factor k is strictly bigger than 1, so when a is small, $C := (k - |a| \frac{\tau}{m_1})^{-1}$ is a contraction factor strictly less than 1, which gives inequality 61. \square

Theorem 11.10 (Contracting map). *There exists a monotonically increasing and right continuous function $h : [0, \infty) \rightarrow [0, \infty)$ such that $h(s) < s$ for each $s > 0$ and*

$$d(F(f), F(g)) \leq h(d(f, g)),$$

for any $f, g \in \mathcal{F}$.

Proof. Let $h : [0, \infty) \rightarrow [0, \infty)$ be

$$h(s) := \sup_{\substack{f, g \in \mathcal{F}, t \in S^1 \\ d(f(2t \times \mathbb{D}_r), g(2t \times \mathbb{D}_r)) \leq s}} d(F \circ f(t \times \mathbb{D}_r), F \circ g(t \times \mathbb{D}_r)).$$

It is easy to see that h is increasing and that $h(0) = 0$. Moreover, by definition

$$d(F(f), F(g)) \leq h(d(f, g)),$$

for any $f, g \in \mathcal{F}$.

By Proposition 11.9 we know that

$$d(F \circ f(t \times \mathbb{D}_r), F \circ g(t \times \mathbb{D}_r)) < C(f, g, t) d(f(2t \times \mathbb{D}_r), g(2t \times \mathbb{D}_r)), \quad (62)$$

where $0 \leq C(f, g, t) < 1$ is the contraction factor. It follows that $h(s) \leq s$ for all $s \geq 0$. This right-hand limit $h(s+) := \lim_{\delta \searrow 0} h(s + \delta)$ exists everywhere since the function h is monotonically increasing. We want to show that $h(s+) < s$ for all $s > 0$.

Suppose that $h(s+) = s$ for some $s > 0$. Let $(\delta_n)_{n \geq 1}$ be a strictly decreasing sequence of positive numbers converging to 0. For each n there exists fibers f_n, g_n and a $t_n \in S^1$ such that

$$d(F \circ f_n(t_n \times \mathbb{D}_r), F \circ g_n(t_n \times \mathbb{D}_r)) > h(s + \delta_n) - \delta_n \quad (63)$$

and where $d(f_n(2t_n \times \mathbb{D}_r), g_n(2t_n \times \mathbb{D}_r)) \leq s + \delta_n$. This follows from the definition of $h(s + \delta_n)$ as a supremum. In view of relation 62 we get that

$$\begin{aligned} h(s + \delta_n) - \delta_n &< d(F \circ f_n(t_n \times \mathbb{D}_r), F \circ g_n(t_n \times \mathbb{D}_r)) \\ &< C_n d(f_n(2t_n \times \mathbb{D}_r), g_n(2t_n \times \mathbb{D}_r)) \leq C_n(s + \delta_n) < s + \delta_n, \end{aligned}$$

where $C_n := C(f_n, g_n, t_n)$ is a number as in Equation 62 above, with $0 \leq C_n < 1$ for every $n \geq 1$. Dividing both sides by $s + \delta_n$ and passing to the limit as $n \rightarrow \infty$ yields

$$\frac{h(s+)}{s} = 1 \leq \lim_{n \rightarrow \infty} C_n \leq 1.$$

Thus $\lim_{n \rightarrow \infty} C_n$ exists and is equal to 1. However, this can only happen if for all $n \geq n_0$ the fibers f_n and g_n belong to the normalizing tubular neighborhood of the semi-parabolic fixed point and the distance between the fibers is measured in the Euclidean metric (in fact the pull-back of the Euclidean metric under the normalizing map). Otherwise, the contraction factor $C(f_n, g_n, t_n)$ is bounded by a uniform constant $K < 1$.

The contraction factor C_n is constructed explicitly in Section 10. It is of the form

$$C_n = \frac{1 - \alpha \mathcal{I}(x_{1,n}, x_{2,n})}{1 - \beta \mathcal{I}(x_{1,n}, x_{2,n})},$$

where α, β are fixed constants with $0 < \beta < \alpha$. The numbers $x_{1,n}$ and $x_{2,n}$ are the x -coordinates of two points that belong to the fibers $f_n(2t_n \times \mathbb{D}_r)$, respectively $g_n(2t_n \times \mathbb{D}_r)$. Recall that $\mathcal{I}(x_{1,n}, x_{2,n}) = \int_0^1 |tx_{1,n} + (1-t)x_{2,n}|^q dt$. If $C_n \rightarrow 1$ then $\mathcal{I}(x_{1,n}, x_{2,n}) \rightarrow 0$. In Lemma 10.4 we showed that $\mathcal{I}(x_{1,n}, x_{2,n}) \geq \frac{1}{2(q+1)} \max(|x_{1,n}|^q, |x_{2,n}|^q)$, so $x_{1,n} \rightarrow 0$ and $x_{2,n} \rightarrow 0$. But then $|x_{1,n} - x_{2,n}| \rightarrow 0$. It follows from Lemma 10.3 and the choice of $x_{1,n}$ and $x_{2,n}$ that $d(F \circ f_n(t_n \times \mathbb{D}_r), F \circ g_n(t_n \times \mathbb{D}_r)) \rightarrow 0$ as $n \rightarrow \infty$.

Passing to the limit in Equation 63 yields $0 \geq h(s+) = s$, thus $s = 0$. Contradiction! Therefore $h(s+) < s$ for all $s > 0$. The function $\tilde{h} : s \mapsto h(s+)$ is continuous from the right and verifies all properties of the function h . With a small abuse of notation we will consider this as the function h from the hypothesis. \square

Theorem 11.11 (Browder [Br]). *Let (X, d) be a complete metric space and suppose $f : X \rightarrow X$ satisfies*

$$d(f(x), f(y)) < h(d(x, y)) \quad \text{for all } x, y \in X,$$

where $h : [0, \infty) \rightarrow [0, \infty)$ is increasing and continuous from the right such that $h(s) < s$ for all $s > 0$. Then f has a unique fixed point x^ and $f^n(x) \rightarrow x^*$ for each $x \in X$.*

Proof. We will follow the proof from [KS]. For a fixed $s > 0$, the sequence $(h^n(s))_{n \geq 0}$ is monotone decreasing (not necessarily strictly) and bounded below, so it has a limit as $n \rightarrow \infty$. Since h is continuous from the right, the sequence converges to a fixed point of h . But 0 is the only fixed point of h , so $h^n(s) \rightarrow 0$ for each $s > 0$.

Let $x_0 \in X$ be fixed and consider $x_n = f^n(x_0)$, $n = 1, 2, \dots$. We can show inductively that $d(x_n, x_{n+1}) < h^n(d(x_0, x_1))$ for all $n \geq 0$. Passing to the limit, we get that

$$0 \leq \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) \leq \lim_{n \rightarrow \infty} h^n(d(x_0, x_1)) = 0.$$

Thus $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. We now show that $(x_n)_{n \geq 1}$ is Cauchy. Let $\epsilon > 0$. Since $\epsilon - h(\epsilon) > 0$, we can choose n large enough so that $d(x_n, x_{n+1}) < \epsilon - h(\epsilon)$. Consider the

ball $B(x_n, \epsilon) = \{x \in X \mid d(x_n, x) < \epsilon\}$ of radius ϵ around x_n . Let $z \in B(x_n, \epsilon)$. Then

$$\begin{aligned} d(x_n, f(z)) &\leq d(x_n, f(x_n)) + d(f(x_n), f(z)) \\ &\leq d(x_n, x_{n+1}) + h(d(x_n, z)) \\ &\leq (\epsilon - h(\epsilon)) + h(\epsilon) = \epsilon. \end{aligned}$$

In the last step, we have used the fact that h is increasing, so $d(x_n, z) < \epsilon$ implies $h(d(x_n, z)) \leq h(\epsilon)$. Therefore $f : B(x_n, \epsilon) \rightarrow B(x_n, \epsilon)$. It follows that $d(x_n, x_{n+m}) < \epsilon$ for all $m \geq 0$. Thus our sequence is Cauchy, hence convergent since X is complete. Let $\lim_{n \rightarrow \infty} f^n(x) = x^* \in X$. Then $f(x^*) = x^*$ since f is continuous. Uniqueness of x^* follows from the contractive condition. \square

A weaker version of this theorem was used in [DH] to prove local connectivity of the Julia set of a parabolic polynomial. Note that for $h(s) = Ks$ with $0 < K < 1$, the theorem reduces to the classical Banach fixed point theorem.

Let $\overline{\mathcal{F}}$ be the completion of the space \mathcal{F} in the d -metric defined above.

Proposition 11.12. *The map $F : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$ has a unique fixed point f^* .*

Proof. The operator F is contracting in the metric defined on $\overline{\mathcal{F}}$. The existence and uniqueness of a fixed point follows from the fixed point Theorem 11.11. \square

The fixed point f^* is a continuous surjection $f^* : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow J^+ \cap \overline{V}$. As defined in Section 7 by \overline{V} we denote the set V together with the local stable manifold $W_{loc}^s(\mathbf{q}_a)$ and its preimage $H^{-1}(W_{loc}^s(\mathbf{q}_a)) \cap B'$, which are both in the boundary of V .

Proposition 11.13. *$Im(f^*) = J^+ \cap \overline{V}$.*

Proof. By Lemma 7.3 we have $J^+ \cap \overline{V} = \bigcap_{n \geq 0} H^{-on}(\overline{V} \cap \overline{U^+})$. By construction, $f_0(t, z) = (\gamma_0(t), z)$, so $f_0(\mathbb{S}^1 \times \mathbb{D}_r)$ is the outer boundary of \overline{V} and is entirely contained in U^+ . Moreover, f^* was obtained as a limit of the functions $f_n : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow \overline{V}$, where $f_n(\mathbb{S}^1 \times \mathbb{D}_r) = H^{-1}(f_{n-1}(\mathbb{S}^1 \times \mathbb{D}_r)) \cap V$, so $f_n(\mathbb{S}^1 \times \mathbb{D}_r)$ is the outer boundary of the set $\bigcap_{0 \leq k \leq n} H^{-ok}(\overline{V} \cap \overline{U^+})$. Hence $Im(f^*) = \bigcap_{n \geq 0} H^{-on}(\overline{V} \cap \overline{U^+})$. \square

Proposition 11.14. *The fixed point $f^* : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow J^+ \cap \overline{V}$ has the form*

$$f^*(t, z) = (\varphi_t(z), z),$$

where $\varphi_t(z)$ is continuous with respect to t , holomorphic with respect to z and a .

Proof. The fixed point $f^*(t, z) = (\varphi_t(z), z)$ is obtained as a uniform limit of the sequence $(f_n)_{n \geq 0}$, where $f_0(t, z) = (\gamma_0(t), z)$ and $f_n(t, z) = F^{on}(f_0)(t, z) = (\varphi_t^n(z), z)$, when $n \geq 1$. Each function f_n is continuous with respect to t and holomorphic with respect to z and a and therefore f^* is also continuous with respect to t and holomorphic in z and a . Notice also that for any $t \in \mathbb{S}^1$, $\varphi_t^n(z)$ is injective when $n \geq 1$, so $\varphi_t(z)$ will either be injective or constant by Hurwitz's theorem. \square

12. THE CONJUGACY

In this section we analyze the properties of the fixed point f^* in more detail and construct the conjugacies to a unique model map.

Consider $f^*(t, z) = (\varphi_t(z), z)$, where $\varphi_t(z)$ is continuous with respect to $t \in \mathbb{S}^1$ and analytic with respect to $z \in \mathbb{D}_r$. Let $\sigma : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow \mathbb{S}^1 \times \mathbb{D}_r$.

$$\sigma(t, z) = (2t, a\varphi_t(z)). \quad (64)$$

For sufficiently small $|a| > 0$ the map σ is well-defined. We will see that is also open and injective. Suppose the semi-parabolic Hénon map is written as in Equation 6. The following theorem is an immediate consequence of our construction.

Theorem 12.1. *Let $p(x) = x^2 + c_0$ be a polynomial with a parabolic fixed point of multiplier $\lambda = e^{2\pi i p/q}$. There exists $\delta > 0$ such that for all parameters $(c, a) \in \mathcal{P}_\lambda$ with $0 < |a| < \delta$ the diagram*

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{D}_r & \xrightarrow{f^*} & J^+ \cap \bar{V} \\ \sigma \downarrow & & \downarrow H_{c,a} \\ \mathbb{S}^1 \times \mathbb{D}_r & \xrightarrow{f^*} & J^+ \cap \bar{V} \end{array}$$

commutes.

Proof. From the definition of f^* , we have that $H \circ f^*(t \times \mathbb{D}_r)$ is compactly contained in $f^*(2t \times \mathbb{D}_r)$. Thus we can write

$$H \circ f^*(t, z) = (p(\varphi_t(z)) + a^2w + az, a\varphi_t(z)) = (\varphi_{2t}(a\varphi_t(z)), a\varphi_t(z)) = f^* \circ \sigma(t, z).$$

The last equality follows from $f^* \circ \sigma(t, z) = f^*(2t, a\varphi_t(z)) = (\varphi_{2t}(a\varphi_t(z)), a\varphi_t(z))$. Therefore f^* semiconjugates H on $J^+ \cap \bar{V}$ to σ on $\mathbb{S}^1 \times \mathbb{D}_r$, as claimed. \square

Lemma 12.2. *We have the following expansion for $\varphi_t(z)$*

$$\varphi_t(z) = \gamma(t) - \frac{1}{p'(\gamma(t))}az + \mathcal{O}(a^2).$$

Proof. Consider the sequence $f_n(t, z) = F^{\circ n}(f_0)(t, z) = (\varphi_t^n(z), z)$, for all $n \geq 1$, and $f_0(t, z) = (\gamma_0(t), z)$. By construction we have that $H \circ f_{n+1}(t \times \mathbb{D}_r)$ is compactly contained in $f_n(2t \times \mathbb{D}_r)$, hence

$$H \circ f_{n+1}(t, z) = (p(\varphi_t^{n+1}(z)) + a^2w + az, a\varphi_t^{n+1}(z)) = (\varphi_{2t}^n(a\varphi_t^{n+1}(z)), a\varphi_t^{n+1}(z))$$

and in particular

$$p(\varphi_t^{n+1}(z)) + a^2w + az = \varphi_{2t}^n(a\varphi_t^{n+1}(z)). \quad (65)$$

Consider the sequence of equipotentials $\gamma_n(t)$ as defined in equation 60. Since the Julia set J_p is connected, $p'(\gamma_n(t))$ does not vanish. Moreover, if p is parabolic, $p'(\gamma(t))$ does not vanish either, where γ is the Charat  dory loop of the parabolic polynomial p . We have the following two relations

$$\begin{aligned} \gamma_{n+1}(t) &= p^{-1}(\gamma_n(2t)) \\ (p^{-1})'(\gamma_n(2t)) &= \frac{1}{p'(\gamma_{n+1}(t))}. \end{aligned}$$

Note that for $n = 0$, $p(\varphi_t^1(z)) + a^2w + az = \gamma_0(2t)$ so for a sufficiently small the following expansion holds

$$\begin{aligned}\varphi_t^1(z) &= p^{-1}(\gamma_0(2t) - az - a^2w) = p^{-1}(\gamma_0(2t)) - (p^{-1})'(\gamma_0(2t))az + \mathcal{O}(a^2) \\ &= \gamma_1(t) - \frac{az}{p'(\gamma_1(t))} + \mathcal{O}(a^2).\end{aligned}$$

We show by induction that for $n \geq 1$

$$\varphi_t^n(z) = \gamma_n(t) - \frac{az}{p'(\gamma_n(t))} + \mathcal{O}(a^2).$$

Indeed, rearranging equation 65 yields

$$\begin{aligned}\varphi_t^{n+1}(z) &= p^{-1}(\varphi_{2t}^n(a\varphi_t^{n+1}(z)) - az - a^2w) \\ &= p^{-1}\left(\gamma_n(2t) - \frac{a^2\varphi_t^{n+1}(z)}{p'(\gamma_n(2t))} - az + \mathcal{O}(a^2)\right) = p^{-1}(\gamma_n(2t) - az + \mathcal{O}(a^2)) \\ &= p^{-1}(\gamma_n(2t)) - (p^{-1})'(\gamma_n(2t))az + \mathcal{O}(a^2) = \gamma_{n+1}(t) - \frac{az}{p'(\gamma_{n+1}(t))} + \mathcal{O}(a^2).\end{aligned}$$

Letting $n \rightarrow \infty$ we get the desired expansion for $\varphi_t(z)$. \square

In fact, since the polynomial p is quadratic, $p'(\gamma(t))$ is just $2\gamma(t)$ in the expansion of $\varphi_t(z)$.

Proposition 12.3. *Let p be hyperbolic or parabolic. For sufficiently small $|a| > 0$ the map σ is open and injective. Also $\sigma(\mathbb{S}^1 \times \mathbb{D}_r) \subset \mathbb{S}^1 \times \mathbb{D}_{|a|r'}$, with $r' < r$.*

Proof. If p be hyperbolic or parabolic then there are no critical points in J_p and there exists $\epsilon > 0$ such that if $\xi_1 \neq \xi_2 \in J_p$ such that $p(\xi_1) = p(\xi_2)$ then $|\xi_1 - \xi_2| > \epsilon$. Thus when p is hyperbolic or parabolic $|\gamma(t) - \gamma(t + 1/2)| > \epsilon$ for $t \in \mathbb{S}^1$. From Lemma 12.2 there exists $M > 0$ such that $|\varphi_t(z) - \gamma(t)| < |a|M$ for all $t \in \mathbb{S}^1$ and $z \in \mathbb{D}_r$. Then for $|a| < \frac{\epsilon}{2M}$ the map σ is injective. It is also open because locally it is a homeomorphism. The Julia set J_p is inside a disk of radius 2 [Bu] so $|\gamma(t)| < 2$ and $|\phi_t(z)| < 2 + |a|M$. Since $r > 3$, we can easily find $r' < r$ such that the image of σ is inside $\mathbb{S}^1 \times \mathbb{D}_{|a|r'}$. \square

Proposition 12.4. *Consider $f^*(t, z) = (\varphi_t(z), z)$ and suppose that $f^*(t_1, z_1) = f^*(t_2, z_2)$ for some $t_1, t_2 \in \mathbb{S}^1$ and $z_1, z_2 \in \mathbb{D}_r$. Then $\varphi_{t_1}(z) = \varphi_{t_2}(z)$ for all $z \in \mathbb{D}_r$.*

Proof. If $f^*(t_1, z_1) = f^*(t_2, z_2)$ then $(\varphi_{t_1}(z_1), z_1) = (\varphi_{t_2}(z_2), z_2)$, hence $z_1 = z_2$ and $\varphi_{t_1}(z_1) = \varphi_{t_2}(z_1)$. Denote by $s : \mathbb{D}_r \rightarrow \mathbb{C}$ the holomorphic function $s(z) = \varphi_{t_1}(z) - \varphi_{t_2}(z)$ and assume that $s(z)$ has an isolated zero of order m at z_1 .

The functions $\varphi_{t_1}(z)$ and respectively $\varphi_{t_2}(z)$ were obtained as the limit of the uniformly convergent sequence of holomorphic functions $\varphi_{t_1}^n(z)$ and respectively $\varphi_{t_2}^n(z)$. By Hurwitz's theorem, there exists $\rho > 0$ such that for sufficiently large $n > n_0$, the function $\varphi_{t_1}^n(z) - \varphi_{t_2}^n(z)$ has exactly m zeros in the disk $|z - z_1| < \rho$. This is a contradiction, since by construction $\varphi_{t_1}^n(z) \neq \varphi_{t_2}^n(z)$ for any $n \geq 0$ and $z \in \mathbb{D}_r$. Hence z_1 cannot be an isolated zero of the function s on \mathbb{D}_r . It follows that s vanishes identically on \mathbb{D}_r and so $\varphi_{t_1}(z) = \varphi_{t_2}(z)$ for all $z \in \mathbb{D}_r$. \square

The fixed point $f^*(t, z) = (\varphi_t(z), z)$ depends on the parameter a . We will use the notation $f_a^*(t, z) = (\varphi_t(z, a), z)$ whenever we want to stress out the dependence on a . Let $\delta > 0$ be chosen as in Theorem 12.1.

Proposition 12.5. *Fix $z \in \mathbb{D}_r$ and $a' \in \mathbb{D}_\delta$ and assume that $\varphi_{t_1}(z, a') = \varphi_{t_2}(z, a')$ for some $t_1, t_2 \in \mathbb{S}^1$. Then $\varphi_{t_1}(z, a) = \varphi_{t_2}(z, a)$ for any a with $|a| < \delta$.*

Proof. Let $s : \mathbb{D}_\delta \rightarrow \mathbb{D}_r$ be the holomorphic function $s(a) = \varphi_{t_1}(z, a) - \varphi_{t_2}(z, a)$. Denote by s_n the holomorphic functions $s_n(a) = \varphi_{t_1}^n(z, a) - \varphi_{t_2}^n(z, a)$.

For any $n \geq 0$, and any a with $|a| < \delta$, we have $\varphi_{t_1}^n(z, a) \neq \varphi_{t_2}^n(z, a)$ by construction. Therefore $s_n(a) \neq 0$ for any $n \geq 0$ and any a with $|a| < \delta$.

The sequence s_n converges uniformly to s on \mathbb{D}_δ . By Hurwitz, s has either no zeros on \mathbb{D}_δ or vanishes identically on \mathbb{D}_δ . Since we know that $s(a_1) = 0$ it follows that s vanishes identically, thus $\varphi_{t_1}(z, a) = \varphi_{t_2}(z, a)$ for any a with $|a| < \delta$. \square

Proposition 12.6. *Consider $t_1 \neq t_2 \in \mathbb{S}^1$. The following statements are equivalent*

- a) $f_a^*(t_1, z) = f_a^*(t_2, z)$ for some a with $|a| < \delta$ and some $z \in \mathbb{D}_r$
- b) $f_a^*(t_1, z) = f_a^*(t_2, z)$ for any $z \in \mathbb{D}_r$ and for any a with $|a| < \delta$
- c) $\gamma(t_1) = \gamma(t_2)$.

Proof. By Propositions 12.4 and 12.5 we know that if $f_a^*(t_1, z) = f_a^*(t_2, z)$ for some a with $|a| < \delta$ and some $z \in \mathbb{D}_r$ then $f_a^*(t_1, z) = f_a^*(t_2, z)$ for any $a \in \mathbb{D}_\delta$ and for any $z \in \mathbb{D}_r$. In particular when $a = 0$ we must have $f_0^*(t_1, z) = f_0^*(t_2, z)$. This is equivalent to $(\gamma(t_1), z) = (\gamma(t_2), z)$, hence $\gamma(t_1) = \gamma(t_2)$. \square

On \mathbb{S}^1 we have a natural equivalence relation \sim_p given by the Thurston lamination of the polynomial p as follows: $t_1 \sim_p t_2$ whenever $\gamma(t_1) = \gamma(t_2)$. Then the set \mathbb{S}^1 / \sim_p is homeomorphic to J_p and the polynomial p acting on its Julia set J_p is topologically conjugate to the angle doubling map on \mathbb{S}^1 / \sim_p as in [Th] and [Th1].

This allows us to determine the equivalence classes of f^* . We define an equivalence relation \sim on $\mathbb{S}^1 \times \mathbb{D}_r$ so that $(t_1, z) \sim (t_2, z)$ whenever $\gamma(t_1) = \gamma(t_2)$. By Lemma 12.2 $\varphi_t(z)$ can be written as

$$\varphi_t(z) = \gamma(t) - \frac{az}{p'(\gamma(t))} + a^2\beta(t, z, a).$$

In view of Proposition 12.6 above, $\beta(t_1, z, a) = \beta(t_2, z, a)$ whenever $\gamma(t_1) = \gamma(t_2)$. Clearly \sim is closed. We would like to identify the quotient space $\mathbb{S}^1 \times \mathbb{D}_r / \sim$ with $J_p \times \mathbb{D}_r$ and the map σ on $\mathbb{S}^1 \times \mathbb{D}_r$ defined in Equation 64 with a similar map σ_p acting on $J_p \times \mathbb{D}_r$.

Consider a map $\sigma_p : J_p \times \mathbb{D}_r \rightarrow J_p \times \mathbb{D}_r$ of the form

$$\sigma_p(\zeta, z) = \left(p(\zeta), a \left(\zeta - \frac{az}{p'(\zeta)} + a^2\beta(\gamma^{-1}(\zeta), z, a) \right) \right). \quad (66)$$

It is well defined, in view of the discussion above.

The map $g : \mathbb{S}^1 \times \mathbb{D}_r / \sim \rightarrow J_p \times \mathbb{D}_r$, $g(t, z) = (\gamma(t), z)$ is a homeomorphism which makes the diagram

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{D}_r / \sim & \xrightarrow{g} & J_p \times \mathbb{D}_r \\ \sigma \downarrow & & \downarrow \sigma_p \\ \mathbb{S}^1 \times \mathbb{D}_r / \sim & \xrightarrow{g} & J_p \times \mathbb{D}_r \end{array} \quad (67)$$

commute. The conjugacy follows directly from the fact that $p(\gamma(t)) = \gamma(2t)$.

The map σ_p on $J_p \times \mathbb{D}_r$ has the form

$$\sigma_p(\zeta, z) = \left(p(\zeta), a\zeta - \frac{a^2 z}{p'(\zeta)} + \mathcal{O}(a^3) \right).$$

and can be further conjugated to a solenoidal map

$$\psi(\zeta, z) = \left(p(\zeta), a\zeta - \frac{a^2 z}{p'(\zeta)} \right).$$

For $|a| > 0$ small enough σ_p and ψ are well-defined, open, and injective. Both maps depend on a and we will use the notation ψ_a to mark the dependence of ψ on a , but we will use ψ when there is no confusion. We will show in Lemma 12.8 that for $0 < |a| < \delta$ all ψ_a are conjugate to each other. Fix ϵ so that $0 < \epsilon < \delta$. Then ψ_a and ψ_ϵ are conjugate and ψ_ϵ does not depend on a .

Lemma 12.7. *There is a homeomorphism $h : J_p \times \mathbb{D}_r \rightarrow J_p \times \mathbb{D}_r$ conjugating σ_p to ψ .*

Proof. We first show that there exists a homeomorphism

$$h : J_p \times \mathbb{D}_r - \sigma_p(J_p \times \mathbb{D}_r) \rightarrow J_p \times \mathbb{D}_r - \psi(J_p \times \mathbb{D}_r)$$

which is the identity on the outer boundary $J_p \times \partial\mathbb{D}_r$ and given by the formula

$$h(\zeta, z) = \psi \circ \sigma_p^{-1}(\zeta, z)$$

on the inner boundary $\sigma_p(J_p \times \partial\mathbb{D}_r)$. Define the space \mathcal{H} of fiber homeomorphisms

$$J_p \times \mathbb{D}_r - \sigma_p(J_p \times \mathbb{D}_r) \rightarrow J_p \times \mathbb{D}_r - \psi(J_p \times \mathbb{D}_r)$$

that agree with h on the boundary as a fiber bundle over J_p . Let $\zeta \in J_p$ and let \mathcal{H}_ζ be the fiber above ζ in \mathcal{H} . We know that $|p'(\zeta)| = 2|\zeta|$ is bounded above and below since J_p does not contain the critical point of p . The fiber above ζ in the range of the homeomorphism h is a disk of radius r with two disjoint disks of radius $\frac{r|a|^2}{2|\zeta|}$ removed, that is

$$\mathbb{D}_r - \bigcup_{\xi \in p^{-1}(\zeta)} \mathbb{D}_{\frac{r|a|^2}{2|\zeta|}}(a\xi).$$

There are d such disks removed if the polynomial has degree d . Similarly, the fiber above ζ in the domain is the disk \mathbb{D}_r with two disjoint simply connected domains removed. These are topological disks of center $a\xi + \mathcal{O}(a^3)$ and radius at most $\frac{r|a|^2}{2|\zeta|} + \mathcal{O}(|a|^3)$, for all $\xi \in p^{-1}(\zeta)$.

In \mathcal{H}_ζ we consider only those fiber homeomorphisms h' which agree with h on the boundary and which move all points by at most $\mathcal{O}(|a|^3)$. Since the term $\mathcal{O}(|a|^3)$ is much smaller compared to $\frac{r|a|^2}{2|\zeta|}$ when a is small, there are no Dehn twists created as

ζ moves on J_p . Therefore all such homeomorphisms are homotopic and this defines a preferred class of homeomorphisms. Note that \mathcal{H}_ζ is not empty. Furthermore, \mathcal{H}_ζ is contractible. This argument is in the same spirit as Lemma 6.8 in [HOV2] and follows from a theorem of Hamstrom [Ham] (which states that if S is a compact surface with nonempty boundary – in our case a disk with two disjoint disks removed – then the components of the group of homeomorphisms which are the identity on the boundary are contractible).

\mathcal{H} is a locally trivial fiber bundle over J_p , with contractible fibers. A fiber bundle with contractible fibers over a paracompact base has a continuous section. Hence there exists a map $s : J_p \rightarrow \mathcal{H}$, $s(\zeta) = h_\zeta$, which associates to each ζ a homeomorphism h_ζ , so that the choice is continuous with respect to ζ . Set h to be s . We now extend h on the inner levels by the dynamics, so we are able to construct a homeomorphism

$$h : J_p \times \mathbb{D}_r - \bigcap_{n \geq 0} \sigma_p^{on}(J_p \times \mathbb{D}_r) \rightarrow J_p \times \mathbb{D}_r - \bigcap_{n \geq 0} \psi^{on}(J_p \times \mathbb{D}_r)$$

which conjugates σ_p to ψ . Furthermore, we extend to the Cantor set (in each fiber) by continuity. \square

Lemma 12.8. *There exists a homeomorphism $h_{a,\epsilon} : J_p \times \mathbb{D}_r \rightarrow J_p \times \mathbb{D}_r$ conjugating ψ_a to ψ_ϵ .*

Proof. We need to consider the space of homeomorphisms \mathcal{H} and construct a preferred class of homeomorphisms. The proof is the same as the proof of Lemma 12.7 above. The same idea was also used in Lemma 5.5 from [R]. \square

Consider the linear change of variables $(\zeta, z) \mapsto (\zeta, az)$. For $|a| > 0$ this conjugates $\psi_a : J_p \times \mathbb{D}_r \rightarrow J_p \times \mathbb{D}_r$ to a map $\psi'_a : J_p \times \mathbb{D}_{r'} \rightarrow J_p \times \mathbb{D}_{r'}$, where $r' = r/|a|$ and

$$\psi'_a(\zeta, z) = \left(p(\zeta), \zeta - \frac{a^2 z}{p'(\zeta)} \right). \quad (68)$$

Similarly ψ_ϵ is conjugate to ψ'_ϵ . Note that all these maps depend on the polynomial p . When p is hyperbolic, Lemma 12.8 is Proposition 6.13 from [HOV2], and as we have seen, the situation is not very different when p is parabolic. The map ψ'_ϵ is the same model map that was used in [HOV2] in understanding Hénon maps that are small perturbations of hyperbolic polynomials.

We now have all the ingredients to complete the proof of the theorems described in the introduction.

Proof of Theorem 1.1. The proof follows directly from Theorem 12.1 and from Lemma 12.7. \square

Based on the construction of the set V from Section 7, the Julia set J of the Hénon map is $J = \bigcap_{n \geq 0} H^{on}(J^+ \cap \bar{V})$. Let $\Sigma := \bigcap_{n \geq 0} \sigma^{on}(\mathbb{S}^1 \times \mathbb{D}_r)$. Then Σ is a (dyadic) solenoid for $0 < |a| < \delta$ and in view of Theorem 12.1, Proposition 12.6 and the above discussion, we can present J as a quotiented solenoid, $J \simeq \Sigma / \sim$. Therefore J is connected.

More directly, we can regard J as

$$J \simeq \bigcap_{n \geq 0} \psi_\epsilon^{\circ n}(J_p \times \mathbb{D}_r).$$

Proof of Theorem 1.2. The proof follows directly from the model of J described above and Corollary 12.9.1 below. \square

Theorem 12.9. *Let $p(x) = x^2 + c_0$ be a polynomial with a parabolic fixed point of multiplier $\lambda = e^{2\pi i p/q}$. There exists $\delta > 0$ such that for all parameters $(c, a) \in \mathcal{P}_\lambda$ with $0 < |a| < \delta$ there exists a homeomorphism g^* which is continuous with respect to ζ and analytic in a and z and which makes the diagram*

$$\begin{array}{ccc} J_p \times \mathbb{D}_r & \xrightarrow{g^*} & J^+ \cap \overline{V} \\ \sigma_p \downarrow & & \downarrow H_{c,a} \\ J_p \times \mathbb{D}_r & \xrightarrow{g^*} & J^+ \cap \overline{V} \end{array}$$

commute.

Proof. The homeomorphism g^* is a composition between f^* and the inverse of the homeomorphism g defined above, in Equation 67. The map σ_p is given in Equation 66. \square

Corollary 12.9.1. The Julia set J equals J^* , the closure of the saddle periodic points.

Proof. The Julia set J is homeomorphic to a quotiented solenoid. Since the periodic points are dense in the solenoid, we get that J is the closure of the periodic points of the Hénon map. Let $x_a \in J$ be a periodic point of period k of the Hénon map H_a , different from the semi-parabolic fixed point \mathbf{q}_a . The periodicity of x_a induces a periodicity on the disks that foliate $J_p \times \mathbb{D}_r$, namely there exists a periodic point $\zeta \in J_p$, $p^{\circ k}(\zeta) = \zeta$ of the parabolic polynomial p such that $x_a \in g^*(\zeta \times \mathbb{D}_r)$ and $\sigma_p^{\circ k}(\zeta \times \mathbb{D}_r)$ is compactly contained inside $\zeta \times \mathbb{D}_r$. Note that $\zeta \neq q_0$, where q_0 is the parabolic fixed point of p . The conjugacy map $g^*(\zeta, z)$ is holomorphic with respect to z , so the stable multipliers of the Hénon map coincide with the stable multipliers of the map

$$\sigma_p(\zeta, z) = \left(p(\zeta), a\zeta - \frac{a^2 z}{p'(\zeta)} + \mathcal{O}(a^3) \right).$$

Let $\lambda^{s/u}$ be the eigenvalues of $DH_{x_a}^{\circ k}$. Then $\lambda^s = \mathcal{O}(a^{2k})$ and $\lambda^u = (p^{\circ n})'(\zeta) + \mathcal{O}(a)$. The function g^* is holomorphic with respect to a , so the disks that foliate $J^+ \cap \overline{V}$ move holomorphically with a . The point x_a moves holomorphically with a and we have $x_a \rightarrow \zeta$ as $a \rightarrow 0$.

The polynomial Julia set J_p is the closure of the repelling periodic points [M]. By the Fatou-Shishikura inequality [S], a polynomial of degree $d \geq 2$ has at most $d - 1$ non-repelling cycles. Since p is quadratic and has a parabolic fixed point q_0 , all other periodic cycles are repelling. Therefore $|(p^{\circ n})'(\zeta)| > 1$. Clearly, when a is small, $|\lambda^u| > 1$ and $|\lambda^s| < 1$, so the periodic point x_a is a saddle point of the Hénon map.

Let δ be as in Theorem 1.1. We show that the periodic point x_a is saddle. It is easy to see that $|\lambda^s| < 1$, so we only need to show that $|\lambda^u| > 1$. Assume that $|\lambda^u| = 1$ for some parameter a_0 with $0 < |a_0| < \delta$. Then we can perturb a_0 so that $|\lambda^u|$ becomes strictly smaller than 1. Otherwise $1/|\lambda^u|$ would have a local maximum at a_0 , which is not possible. Thus we can find a parameter a close to a_0 for which x_a is a sink, and as such it must belong to the interior of K^+ and not to J^+ ; contradiction. It follows that all periodic points are saddles, except the semi-parabolic fixed point, hence $J = J^*$. \square

13. CONCLUSIONS

Let \mathcal{P}_λ^n be the set of parameters $(c, a) \in \mathbb{C}^2$ for which the Hénon map $H_{c,a}$ has a cycle of period n with one multiplier λ a root of unity. In other words, the n^{th} iterate $H_{c,a}^{\circ n}$ of the Hénon map has a fixed point \mathbf{q} such that its derivative $DH_{c,a}^{\circ n}(\mathbf{q})$ has eigenvalues λ and $\mu = (-1)^n a^{2n}/\lambda$.

When $n = 1$ the curve \mathcal{P}_λ^1 is just the curve \mathcal{P}_λ from Equation 2 which has a nice global characterization. We also have a nice description for $n = 2$. However, when $n \geq 3$, it is hard to give an explicit formula for the curve \mathcal{P}_λ^n . In any case, \mathcal{P}_λ^n is an algebraic set that intersects the parametric line $a = 0$ in a discrete set of points. Suppose that $(c_0, 0)$ is such a point of intersection, then $c = c(a)$ is locally a function of the parameter a when $(c, a) \in \mathcal{P}_\lambda^n$ and $|a| < \delta$ is sufficiently small. The results stated in Section 1 hold for small perturbations inside the curve \mathcal{P}_λ^n of the quadratic polynomial $x \mapsto x^2 + c_0$ with a parabolic n -cycle of multiplier λ .

The technique presented in this paper and the results from Section 1 can also be easily generalized to Hénon maps that are small perturbations (inside appropriate algebraic sets analogous to the curves \mathcal{P}_λ^n) of a polynomial p of degree $d \geq 2$ whose critical points are attracted either to attractive or parabolic fixed points (or cycles). The local model space for J^+ is $J_p \times \mathbb{D}_r$ and the model map acting on it is $\psi(\xi, z) = \left(p(\xi), a\xi - \frac{a^2 z}{p'(\xi)} \right)$. The map ψ is again well defined because the Julia set J_p contains no critical points.

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